University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 1: Classical Mechanics

09 January 2023, 1:00 pm - 4:00 pm

General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 180 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
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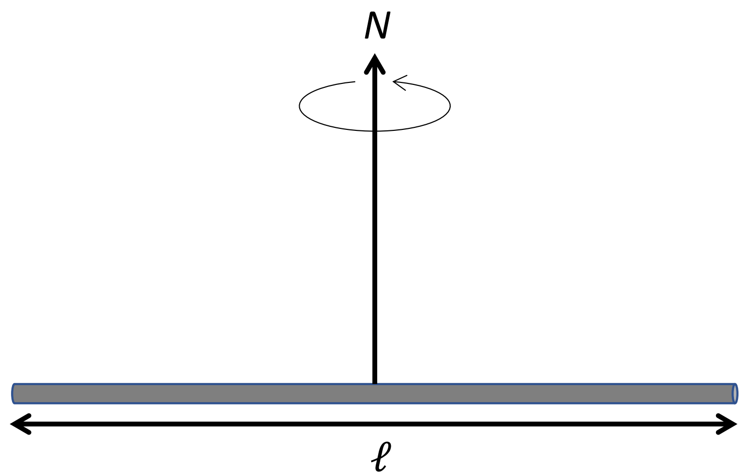
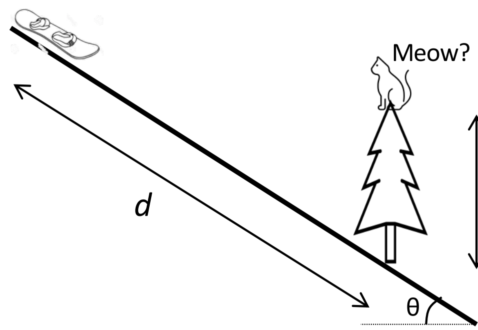
1. Consider a thin rod of mass m and length ℓ at rest.



(a) (**10 points**) Calculate the moment of inertia of the rod about its center.

(b) (**10 points**) At $t = 0$, the rod is subject to a constant torque of magnitude N. How long will it take for it to spin up to an angular velocity Ω?

2. Consider a snow-covered hill that makes an angle θ to the horizontal, on which sits a snowboard of mass m_{sb}. At a distance d down the hill is a tree of height h, at the top of which is a cat. The coefficient of kinetic friction of the snowboard with the snow is μ.



At $t = 0$, the snowboard begins to slide down the hill under the influence of gravity.

(a) (**12 points**) The cat wants to drop out of the tree and land perfectly on the snowboard as it passes the base of the tree below it. How long must the cat wait after $t = 0$ before it jumps?

(b) (**5 points**) There is a minimum value of d for which there is a physical solution. What is this minimum value of d?

(c) (**3 points**) Explain the previous answer – describe **in words** what happens for values of d less than that.

3. On the Earth, consider a bead of mass m on a frictionless wire that traces a parabola: its height y is related to the horizontal displacement x by

$$y = kx^2.$$

   (a) (**7 points**) Write down the Lagrangian L in terms of x and $\dot{x}$.

   (b) (**7 points**) Derive the equation of motion for x from the Lagrangian.

   (c) (**6 points**) For small oscillations, where only terms that are linear in x, $\dot{x}$, and $\ddot{x}$ matter, what is the frequency of oscillations about the origin?

4. Consider a particle of mass m moving on xy-plane in the central potential, $V(r) = -\frac{\alpha}{r}$, where $r = \sqrt{x^2 + y^2}$ for the position of the particle (x, y) in the Cartesian coordinates, and α is a positive constant.

   (a) (**4 points**) Write down the Lagrangian of the system in the polar coordinates, r and θ.

   (b) (**4 points**) Determine the Euler-Lagrange equations in the polar coordinates, and show that the angular momentum $\ell \equiv mr^2 \frac{d\theta}{dt}$ is constant in this system.

   (c) (**4 points**) Using the angular momentum conservation, we can express the Euler-Lagrange equation for r of the form,

   $$m\frac{d^2r}{dt^2} = -\frac{d}{dr}V_{\text{eff}}(r),$$

   where V_{eff} is the so-called effective potential. Express the effective potential in terms of α, r, m, and ℓ.

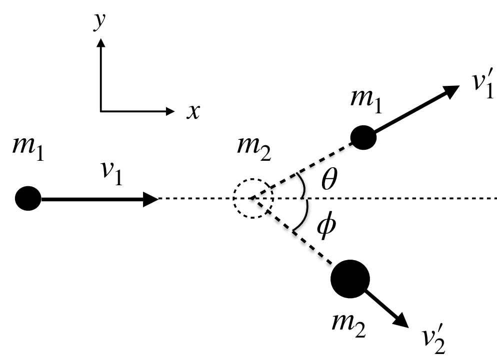
   (d) (**2 points**) If we define the energy of the system as

   $$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V_{\text{eff}}(r),$$

   show that the this energy is conserved.

   (e) (**6 points**) For a fixed value of ℓ and certain initial conditions, the particle undergoes uniform circular motion. Express the radius, the energy, and the period of this motion in terms of α, m, and ℓ.

5. Consider the elastic collision between two particles on xy-plane, neglecting any external forces on the system consisting of two particles. Particle 1 of mass m_1 is initially moving along the positive x direction with speed v_1 and collides elastically with particle 2 of mass m_2 initially at rest. After the collision, particles 1 and 2 move with speeds v_1' and v_2' in the directions specified by the scattering angles θ and ϕ, respectively, as shown in the figure.



(a) (**6 points**) Considering the momentum conservation of the system, derive the following relations:

$$v_1' = \frac{\sin\phi}{\sin(\theta+\phi)} v_1, \quad v_2' = \frac{\sin\theta}{\sin(\theta+\phi)} \frac{m_1}{m_2} v_1.$$

(b) (**6 points**) Consider the energy conservation of the system, and derive the relation,

$$\frac{m_1}{m_2} = \frac{\sin^2(\theta+\phi) - \sin^2\phi}{\sin^2\theta} = \frac{\sin(\theta+2\phi)}{\sin\theta}.$$

(c) (**4 points**) If $m_1 = m_2$ and $\theta = \frac{\pi}{6}$, find ϕ.

(d) (**4 points**) If $m_1 \ll m_2$ and $\theta = \frac{\pi}{6}$, find ϕ.

6. Consider one dimensional motion of a particle with mass m in a potential $U(x)$, where $x(t)$ is the position of the particle. Suppose this particle undergoes oscillatory motion between x_A and x_B ($x_A < x_B$). The Lagrangian of the system is given by

$$L = \frac{1}{2}m\dot{x}^2 - U(x),$$

where $\dot{x} \equiv \frac{dx}{dt}$.

(a) (**2 points**) Write the equation of motion for the particle.

(b) (**2 points**) Write the total energy of the system (E) and show that it is conserved.

(c) (**4 points**) Considering the energy conservation for the oscillatory motion, find the relation among E, $U(x_A)$, and $U(x_B)$.

(d) (**6 points**) The formula of the total energy of the system, which is constant, is the first-order differential equation. By integrating it out, derive the formula for the period of the oscillation, i.e., twice the time during which the particle passes from x_A to x_B.

(e) (**6 points**) As is well known, if $U(x) = \frac{1}{2}m\omega^2 x^2$ with a positive constant ω, the system is the harmonic oscillator, and the period of the oscillation is given by $T = 2\pi/\omega$. Obtain this result by using the formula found in (d). One may set $x_A = -1$ and $x_B = 1$ in this calculation and use the formula $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$.

University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 2: Electromagnetism

10 January 2023, 1 pm - 4 pm

General Instructions

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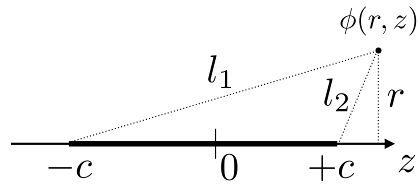
1. A uniformly charged rod of length $2c$ and total charge q is oriented along the z-axis, and centered at the origin (see figure). Let r be the radial coordinate perpendicular to the z-axis.



   (a) (**10 points**) Find the electric potential $\phi(r, z)$ as a function of the z and r coordinates. Express your result with the help of $l_1 = \sqrt{r^2 + (z + c)^2}$ and $l_2 = \sqrt{r^2 + (z - c)^2}$.
   HINT: $\int \frac{dx}{\sqrt{a^2+x^2}} = \log \frac{\sqrt{a^2+x^2}+x}{a}$.

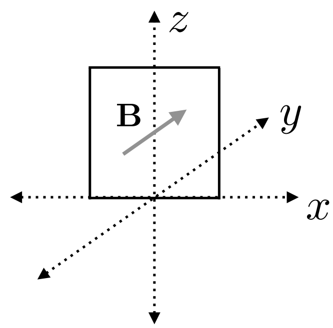
   (b) (**7 points**) Show that the potential can be written as

$$\phi = \frac{1}{4\pi\varepsilon_0} \frac{q}{2c} \log \frac{u + c}{u - c}, \tag{1}$$

   in elliptical coordinates u and v, for which $l_1 = u + v$, and $l_2 = u - v$.
   HINT: first show that $uv = cz$.

   (c) (**3 points**) Starting from Eq. (1), find an approximate expression for ϕ in the limit $u \gg c$, and interpret the result.

2. A quadratic loop with side length a and mass m is placed in a magnetic field $\mathbf{B}$ (see figure). The loop is made of a conducting material and has resistance R. The magnetic field points in the y-direction, and is finite for $z > 0$ while it vanishes for $z \leq 0$. At $t = 0$, the loop lies in the x-z plane, the lower edge of the loop is placed on the x-axis at $z = 0$ and loop starts falling in the $-z$ direction under the influence of the gravitational force.



$\mathbf{B} \neq 0$ for $z > 0$

$\mathbf{B} = 0$ for $z \leq 0$

(a) **(12 points)** Find the differential equation that describes the time-dependence of the velocity $\mathbf{v}$ of the loop as long as it is partially inside the magnetic field. Explain your reasoning.

(b) **(8 points)** Find $\mathbf{v}(t)$ under the conditions specified in (a).

3. **(20 points)** Find the continuity equation for momentum in electromagnetism including both matter and field contributions. Express your result in the form,

$$\partial_t(P_k^{mech} + P_k^{field}) = \int d^3r \sum_{i=1}^{3} \nabla_i T_{ik}, \tag{2}$$

and provide expressions for the field contribution to the momentum $\mathbf{P}^{field}$ and the components of the energy-momentum tensor T. HINT: Start by expressing $\partial_t \mathbf{P}^{mech}$ through the Lorentz force density $\mathbf{f} = \rho\mathbf{E} + c^{-1}\mathbf{j} \times \mathbf{B}$ and subsequently use Maxwell equations to arrive at the continuity equation. To symmetrize your result, you may use the relation $\nabla \cdot \mathbf{B} = 0$. Finally, the result for the energy-momentum tensor can be written in a compact form with the help of the relation

$$[\mathbf{X}(\nabla \cdot \mathbf{X}) - \mathbf{X} \times (\nabla \times \mathbf{X})]_j = \sum_{i=1}^{3} \partial_i \left[X_i X_j - \frac{1}{2}\delta_{ij} \mathbf{X} \cdot \mathbf{X} \right]. \tag{3}$$

4. (**20 points**) **Figure 01** shows two curved plastic rods, one of charge q and the other of charge $-q$. They are brought together to form a circle of radius R in a xy-plane. The x-axis passes through both of the connecting points, and the charge is distributed uniformly on both rods. What are the (a) magnitude and (b) direction (relative to the positive direction of the x-axis) of the electric field produced at P, the center of the circle?

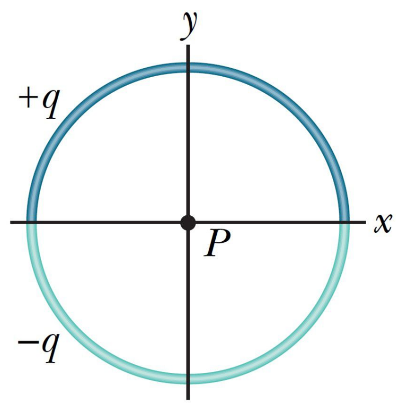


Figure 01

5. (**20 points**) **Figure 02** shows, in cross section, two solid spheres (labeled 1 and 2) with volume charge densities $\rho_1 = Ar^4$ and $\rho_2 = Br$, where A and B are constants. Each has radius R. Point P lies on a line connecting the centers of the spheres, at radial distance $R/2$ from the center of sphere 1. If the net electric field at point P is zero, what is the ratio A/B?

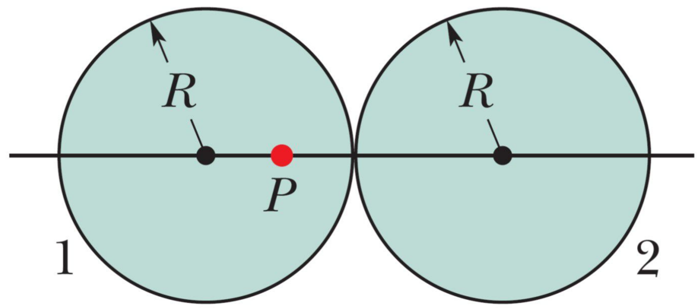


Figure 02

6. (**20 points**) In **Figure 03**, a rectangular loop of wire with length a, width b, and resistance R is placed near an infinitely long wire carrying current i. The loop is then moved away from the wire at constant speed v. When the center of the loop is at distance $r = \frac{3b}{2}$, what are **(a)** the magnitude of the magnetic flux through the loop and **(b)** the current induced in the loop?

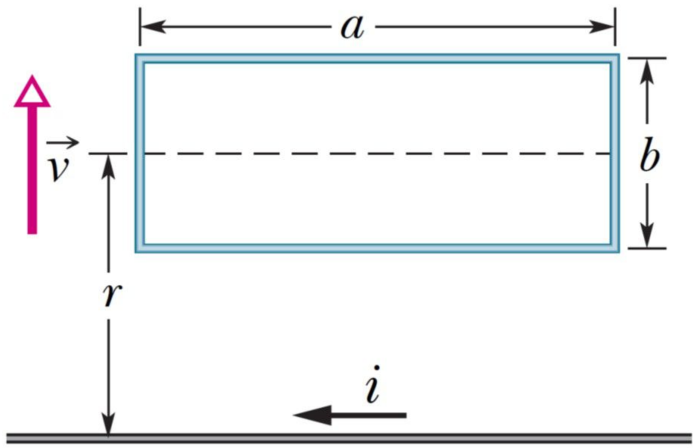


Figure 03