University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 3: Thermal Physics

22 August 2023, 3:00 pm - 6:00 pm

General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 180 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. Consider a system of N localized spins with spin $1/2$, subject to a homogeneous magnetic field $\mathbf{B}$. The energy of the system is given by

$$E = -(N_\uparrow - N_\downarrow)\mu_B B = -M\mu_B B,$$

where μ_B is the Bohr magneton and $N_\uparrow$ ($N_\downarrow$) denotes the number of spins parallel (antiparallel) to $\mathbf{B}$.

   (a) (12 pts) Calculate the entropy S of the system as a function of N and M in the micro-canonical ensemble.

   (b) (8 pts) Find the inverse temperature $1/T$ from the result of (a).

2. (a) (10 pts) Calculate the canonical partition function of a system of N independent quantum harmonic oscillators with the same frequency ω.

   (b) (10 pts) Consider a classical ideal gas in an infinitely high cylindrical container. A homogeneous gravitational field acts in the direction of the cylinder axis. Calculate the average kinetic energy per particle of the gas by means of the canonical ensemble. [Note $\int_{-\infty}^{\infty} dx \mathrm{e}^{-ax^2} = \sqrt{\pi/a}$].

3. (a) (8 pts) Using the expression for the grand canonical potential of an ideal Bose gas, $\Omega(T, V, \mu) = k_B T \sum_r \ln(1 - \mathrm{e}^{-\beta(\varepsilon_r - \mu)})$, calculate the entropy $S(T, V, \mu)$.

   (b) (10 pts) Express $S(T, V, \mu)$ purely as a function of the average occupation number n_r.

   (c) (2 pts) Check the behavior of the entropy for $T \to 0$ for the case of a fixed number N of bosons.

4. (20 pts) Show that for a system constituted of a single component (*e.g.* a single type of atom or molecule), where the product PV^γ (P stands for pressure and V stands for volume) is constant in an adiabatic process ($\gamma > 0$), the internal energy (U) is given by:

$$U = \frac{1}{\gamma - 1} PV + Nf\left(\frac{PV^\gamma}{N^\gamma}\right),$$

where N is the number of components, and $f\left(\frac{PV^\gamma}{N^\gamma}\right)$ is an arbitrary function.

5. (20 pts) Find the fundamental equation ($S = S(U, V, N)$) of a given system that obeys the following equations:

$$U = \frac{1}{2} PV$$

and

$$T^2 = \frac{AU^{3/2}}{VN^{1/2}}$$

where A is a positive constant, P stands for pressure, V stands for volume, T stands for temperature, N stands for number of particles and U stands for energy.

6. The Brayton or Joule cycle consists of two isentropic and two isobaric steps. In a working engine air (and fuel) is compressed adiabatically (A $\to$ B), heated by fuel combustion at constant pressure (B $\to$ C), expanded (C $\to$ D), and rejected to the atmosphere. The process (D $\to$ A) occurs outside the engine, and a fresh charge of air is taken in to repeat the cycle.

   (a) (10 pts) Sketch the Entropy-Pressure diagram for this cycle.

   (b) (10 pts) If the working gas is an monoatomic ideal gas, show that the engine efficiency is given by:

$$\varepsilon = \frac{W_{net}}{Q_{in}} = 1 - \left(\frac{P_A}{P_B}\right)^{\frac{C_P - C_V}{C_P}} .$$

   Remember that the engine efficiency is defined as (net work)/(total heat in), where the (total heat in) includes only the parts of the cycle in which heat flows into the gas and excludes parts in which heat flows out.

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Graduate Qualifying Exam

Part 4: Quantum Mechanics

23 August 2023, 3:00 pm - 6:00 pm

General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 180 minutes are allocated for this exam.
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- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
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1. Consider a 2-state quantum system consisting of a ground state $\mid 0\rangle$ with energy $E_0 = 0$ and an excited state $\mid 1\rangle$ with energy $E_1 = \epsilon$. This system can be observed with an operator A, which has nondegenerate eigenvalues α and β with eigenstates:

$$\mid \alpha\rangle = \frac{1}{\sqrt{2}} \mid 0\rangle + \frac{i}{\sqrt{2}} \mid 1\rangle$$

$$\mid \beta\rangle = \frac{1}{\sqrt{2}} \mid 0\rangle - \frac{i}{\sqrt{2}} \mid 1\rangle$$

   (a) (16 pts) At time $t = 0$, the system is measured using operator A to have a value α. At some later time t, what is the probability that it will be measured to have value β?

   (b) (4 pts) Demonstrate that your answer to part (a) makes physical sense, i.e. it makes sense at $t = 0$ and has reasonable bounds.

2. The simple 1D harmonic oscillator has a Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

The energy and wavefunction of the ground state are:

$$E_0 = \frac{1}{2}\hbar\omega \qquad \varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

Now imagine that we add the following small perturbation ($\epsilon \ll 1$) to the Hamiltonian

$$H = H_0 + \frac{1}{2}\epsilon m\omega^2 x^2$$

(a) (15 pts) Using first order perturbation theory, find the new energy of the ground state.
Note: $\int_{-\infty}^{\infty} x^2\, e^{-x^2}\, dx = \sqrt{\pi}/2$.

(b) (5 pts) This is exactly equivalent to a harmonic oscillator with a slightly different frequency ω'. Find the exact solution ground state energy for this new frequency, and show that if you do a first order Taylor expansion, it agrees with what you found in part (a).

3. For a particle of mass m in free space, the Hamiltonian is

$$H = \frac{P^2}{2m}$$

where the momentum operator

$$P = -i\hbar\nabla.$$

Consider an infinite plane wave moving in the x direction:

$$\psi(\vec{r}, t) = Ce^{i(kx-\omega t)}$$

(a) (7 pts) Prove that $\psi(\vec{r}, t)$ satisfies the time-independent Schrödinger equation

$$H\psi = E\psi.$$

What is the value of the eigenvalue E?

(b) (7 pts) Prove that $\psi(\vec{r}, t)$ satisfies the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi$$

for an appropriate value of ω.

(c) (6 pts) Despite the fact that $\psi(\vec{r}, t)$ satisfies the Schrödinger equation, it is **not** a physically-valid solution. Prove that it does not satisfy one key condition for a valid wavefunction.

4. Consider a one-dimensional Quantum Mechanical system for a particle with mass m. The potential of the system is given by $V(x) = -V_0\,\delta(x)$, where $V_0 > 0$ is a constant, and $\delta(x)$ is the Dirac delta function which has the fundamental property: $\delta(x) = 0$ for $x \neq 0$, and $\int_{-\epsilon}^{\epsilon} dx\, f(x)\,\delta(x) = f(0)$ for $\epsilon > 0$ and a regular function $f(x)$. The time-independent Schrödinger equation of this system is given by

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x),$$

where E is the energy of the state $\psi(x)$. We are interested in a bound state of the system, so that $E = -|E| < 0$.

(a) (**6 points**) For $x \neq 0$, show that the wave function of the grand state (which is an even function of x) can be expressed as

$$\psi(x) = C\,e^{-k|x|},$$

where $k = \frac{\sqrt{2m|E|}}{\hbar}$, and $C = \sqrt{k}$ is the normalization constant.

(b) (**6 points**) Integrating the Schrödinger equation above for the range of $-\epsilon \leq x \leq \epsilon$, and taking the limit of $\epsilon \to 0$, show that the energy of the state is given by

$$E = -\frac{mV_0^2}{2\hbar^2}.$$

(c) (**8 points**) For the grand state, calculate $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ and $\langle(\Delta p^2)\rangle = \langle p^2\rangle - \langle p\rangle^2$, where p is the momentum of the particle, and find $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{2}$. One may use $\int_0^\infty dx\, x^2 e^{-ax} = \frac{2}{a^3}$ for a real and positive parameter a.

5. In quantum mechanics, the angular momentum operator is defined as a vector operator $\mathbf{J} = (J_x, J_y, J_z)$ whose components satisfy the following commutation relations:

$$[J_x, J_y] = i\hbar J_z, \quad [J_y, J_z] = i\hbar J_x, \quad [J_z, J_x] = i\hbar J_y.$$

(a) (**2 points**) The orbital angular momentum operator is defined as $\mathbf{L} = (L_x, L_y, L_z) = \vec{x} \times \vec{p}$, where $\vec{p} = \frac{\hbar}{i}\nabla$. The operators $L_{x,y,z}$ satisfy the same commutation relations as $J_{x,y,z}$. By using the definition of $L_{x,y,z}$, verify $[L_x, L_y] = i\hbar L_z$.

(b) (**2 points**) The spin operators are defined as

$$S_x = \frac{\hbar}{2}\sigma_x, \quad S_y = \frac{\hbar}{2}\sigma_y, \quad S_z = \frac{\hbar}{2}\sigma_z,$$

where $\sigma_{x,y,z}$ are Pauli matrices given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These operators also satisfy the same commutation relations as $J_{x,y,z}$. By using the above matrix form, verify $[S_x, S_y] = i\hbar S_z$.

(c) (**4 points**) Define the operators,J^2 and $J_\pm$, by $J^2 = J_x^2 + J_y^2 + J_z^2$ and $J_\pm = J_x \pm iJ_y$, respectively. They satisfy the following commutation relations: $[J^2, J_z] = 0$, $[J^2, J_\pm] = 0$, $[J_z, J_\pm] = \pm\hbar J_\pm$, and $J_\mp J_\pm = J^2 - J_z(J_z \pm \hbar)$, which are useful in solving the following questions. Verify two of these relations: $[J^2, J_z] = 0$ and $[J_z, J_\pm] = \pm\hbar J_\pm$.

(d) (**2 points**) Suppose $|j, m\rangle$ is an eigenstate of J^2 and J_z, satisfying $J^2|j, m\rangle = j(j+1)\,\hbar^2|j, m\rangle$ and $J_z|j, m\rangle = m\,\hbar\,|j, m\rangle$ with eigenvalues, $j(j+1)\,\hbar^2$ and $m\,\hbar$, respectively. Show that $J_\pm|j, m\rangle \propto |j, m \pm 1\rangle$.

(e) (**6 points**) Using the relation $J_\pm = J_\mp^\dagger$, show $||J_\pm|j, m\rangle|^2 = \langle j, m|J_\mp J_\pm|j, m\rangle = (j \mp m)(j \pm m + 1)\hbar^2||j, m\rangle|^2$. Explain that this formula along with $J_\pm|j, m\rangle \propto |j, m \pm 1\rangle$ indicates that $-j \leq m \leq j$ and j must be an integer or half integer.

(f) (**4 points**) Consider a composite state consist of two particles, A with spin 1/2 and B with spin 1. Classify the composite state with its total spin. Here, we do not consider the orbital angular momentum.

6. Let us derive the hydrogen atom energy spectrum by the quantization condition introduced by Niels Bohr.

   (a) (**4 points**) We model the motion of an electron in hydrogen atom to be a uniform circular motion with radius r under the centripetal Coulomb force, $F = k_e \frac{e^2}{r^2}$, where e is the electric charge of proton, and k_e is the Coulomb constant. Applying the classical mechanics, show that the speed of electron v is given as $v = \sqrt{\frac{k_e e^2}{m_e r}}$, where m_e is the electron mass.

   (b) (**4 points**) We impose the Bohr's quantization condition, $L = n\,A$, where L is the angular momentum of the electron, $n = 1, 2, 3, \cdots$ is an integer, and A is a certain constant. Combining this condition with the electron speed found in (a), derive the following formulas for quantized r and v:

$$r = \frac{n^2 A^2}{k_e e^2 m_e}, \quad v = \frac{k_e e^2}{nA}.$$

   (c) (**4 points**) By using the formulas found in (b), show that the hydrogen atom energy spectrum is given by

$$E_n = -\frac{k_e^2 e^4 m_e}{2n^2 A^2}.$$

   (d) (**4 points**) The energy of photon emitted by the transition from the energy level E_n to E_{n-1} is given by $\Delta E = h\nu_\gamma$, where ν_γ is the frequency of the emitted photon. For $n \gg 1$, show that this frequency is approximately give by

$$\nu_\gamma \simeq \frac{k_e^2 e^4 m_e}{hA^2 n^3}.$$

   (e) (**4 points**) The frequency found in (d) may be identified as the frequency of the electron (ν_e) in the energy level E_n. Determine A by substituting the formulas for r and v found in (b) into the classical mechanics expression of ν_e and comparing it with ν_γ found in (d).