Computing Feynman Integrals via Linear Algebra

References:
1. arxiv: 2201.11657 hep-th (Lin. Alg.)
2. arxiv: hep-ph/0102033 (Difference equ. algorithm)
4. arxiv: 2201.033595 (Write-up of a course on FI's, very nice!)

Outline:
- General Feynman Integral approach (identities + MIs)
- Auxiliary Mass Flow method
- Linear Algebra approach

Generalities of Feynman Integrals
- Higher-loop Feynman Integrals (FI) are complicated and hard to compute
- There are systematic methods to compute them
  - Direct methods (sector decomp, Mellin-Barnes representation, loop-tree duality, etc...)
  - Indirect methods (Difference equations, DPP Eqs.)
- Unfortunately, you need B.C.s, so direct integration is still required.

Generalities
- $N_k$ - # of legs
- $N_e$ - # ext. lines
- $N_i$ - # int. lines
- $N_p = N_e - 1$

- Propagator Denominator:
  $$D_i = q_i^2 - m_i^2$$
• A generic FI in D-Dim Euclidean k-space is

\[ \int [d^{D}k_1] \cdots [d^{D}k_{N_{\delta}}] V_{\gamma}\delta \]

where \( [d^{D}k] = \frac{d^{D}k}{\pi^{D/2}} \)

generic integrand \( V_{\gamma}\delta = \frac{\prod_{i=1}^{N_{\delta}} \prod_{j=1}^{N_{\delta}} (p_i \cdot k_j)^{S_{ij}} \prod_{i=1}^{N_{\delta}} \prod_{j=1}^{N_{\delta}} (k_i \cdot k_j)^{S_{ij}^{2}}} {\prod_{i=1}^{N_{\delta}} D_i^{x_i}} \) (1)

\( (\gamma, x, S_{ij} \geq 0), \gamma = \gamma_1, \ldots, \gamma_{N_{\delta}}, \delta = \delta_{ij} \)

- The numerator of (1) is a product of powers of all possible scalar products involving the loop momenta \( k \).

> total \# of products is \( N_{sp} = N_p N_k + N_k (N_k + 1)/2 \)

• We can obtain algebraic and integration-by-parts identities.

- For each denominator in a generic \( V_{\gamma}\delta \), an identity

\[ \frac{(p \cdot k)^{2}}{D_{i}} = \frac{1}{C_{j}} \left( 1 - \frac{D_{j} - C_{j} (p \cdot k)^{2}}{D_{i}} \right), \quad j = 1, \ldots, N_{\delta} \] (2)

where \( (p \cdot k)_{j} \subset D_{i} \), and \( C_{j} \) is the coefficient of \( (p \cdot k)^{2} \)

in \( D_{j} \). Inner products must be all chosen differently.

Ex: \( D_{j} = (q_{j}^{2} + m_{j}^{2}) = (q_{j}^{2} + (k_{j} \cdot x_{j})^{2} + m_{j}^{2}) = (p^{2} + k^{2} + 2(p \cdot k)_{j} + m_{j}^{2}) \)

\[ C_{j} (p \cdot k)_{j} \]
- Applying these identities in sequence to \( V_{ij\alpha\beta} \) as many times as necessary turns it into a sum of new terms not containing \((p,k)_j\) and \(D_i\) simultaneously:

\[
V'_{ni\alpha\beta} = \frac{\prod_{j=1}^{N_p-n} (p_k \text{ irreducible})^\delta_i}{\prod_{j=1}^{n} D_i^\delta_j}, \quad n \leq N_k, \quad \alpha_i, \beta_i > 0 \quad (3)
\]

> The subscript shows dependence on \( n \rightarrow \delta \) denominators, their specific combination \( i = \delta_1, \ldots, \delta_n \), and the exponents \( \alpha = \delta_1^\alpha_1 \ldots, \delta_n^\alpha_n \), \( \beta = \delta_1^\beta_1 \ldots, \delta_n^\beta_n \).

> \((p,k \text{ irreducible})^\delta_j \); \( j = 1, \ldots, N_p - n \) corresponds to scalar products which can be simplified further with (2).

- Integrating by parts in \( D \) dimensions yields a set of identities:

\[
\int [d^p k_1] \ldots [d^p k_N^e] \frac{2}{\partial (k_{i\mu})} \left( (P_\mu)_{\nu} V'_{ni\alpha\beta} \right) = 0,
\]

\[
j = 1, \ldots, N_k, \quad \nu = 1, \ldots, N_p \quad (4)
\]

\[
\int [d^p k_1] \ldots [d^p k_N^e] \frac{2}{\partial (k_{i\mu})} \left( (k_{\mu})_{\nu} V'_{ni\alpha\beta} \right) = 0,
\]

\[
j = 1, \ldots, N_k
\]

> Each \( V'_{ni\alpha\beta} \) yields \( N_k (N_p + N_k) \) identities.

> Each \( V'_{ni\alpha\beta} \) contains irreducible scalar products.

Calculating derivatives introduces more that can be further reduced via (2).
- Identities will contain linear combinations of two types of integrals:

(i) Ones containing all n denominators \(\{D_{i_1}, \ldots, D_{i_n}\}\)

(ii) Ones missing a single denominator due to algebraic identities.

i.e., each identity will be a linear combination of integrals like

\[
\int [D^{P_1}] \cdots [D^{P_{N+1}}] V^{\nu_1 R_3}
\]

with polynomials of degree \(0\) or \(1\) in the number of dimensions \(D\) as coefficients.

- Generic sets of these identities form a homogeneous linear system of eqns. with the integrals as the unknowns.

\[ \Rightarrow \text{such systems are under-determined} \]

\[ \Rightarrow \exists \ "\text{Master integrals}" (MI) whose values cannot be determined from the system. \]

\[
\Rightarrow \text{Generally, FIs can be grouped into families of FIs:}
\]

\[
I^\nu = \int \left( \prod_{i=1}^{\nu} \frac{d^{P_i} {k_i}}{i \pi^{D/2}} \right) \frac{D^{-\nu_{k_1}} \cdots D^{-\nu_{k_N}}}{D_{i_1}^{\nu_{i_1}} \cdots D_{i_N}^{\nu_{i_N}}} \quad \nu_{1}, \nu_{N} \in \mathbb{Z}
\]

for various values of \(\nu\).

FIs form a finite-dim linear space.
Any FI in a given family can be decomposed into a linear combination of MIs, which forms a finite basis of the linear space formed by the FIs.

- Can use integration by parts (IBP) reduction to find MIs for multiloop FIs.
- Ref 2 uses a method using a finite # of identities using carefully chosen parameters \( n, \alpha, \beta \) from a large finite set.

If \( \sum_{i} c_i W_j = 0 \) is an identity, one can re-write some of the integrals \( W_j \) in terms of other integrals and substitute into the identity, which becomes \( \sum_{j} c_j W_j = 0 \).

Choose a specific integral \( W_e \) and write \( W_e = \sum_{j} c_j W_j \), then one can add the new integral to the system of substitutions in \( W_e \).

It's a complicated algorithm, so I can't write it here, but any integral can eventually be written as

\[
I = \sum_{i} \text{re}\, B_e
\]

where \( \text{re} \) are rational \( \mathbb{Q} \) or D neurons, scalar products or external momenta, and

\( B_e \) are master integrals

\[
B_e = \int [D^0_k]...[D^{p k_m}] \prod_{i=1}^{16} (p.k.\mod{p.m}) \frac{d^{p, i}}{D_i \, D_{i'}...D_{i''}}
\]

where the combo of indices \( i \) & exponents \( p \) depend on the index \( i \), and the \( p \)-dependence is given in the master integrals from alg.
Since FI's can always be written in terms of MI's using IBP reduction, only MI's need to be studied.

- FI's containing linear propagators can also be determined by FI's containing only quadratic propagators (see reference).

- Auxiliary Mass Flow (AMF) method

- It's possible to compute MI's systematically using differential equation method - rewriting the derivative of an MI with its kinematic variables in terms of a linear combination of other MI's using other reduction methods.

- Still needs Boundary Conditions. Since no general rules to find good BC's, doing this systematically is difficult.

- Consider family of dimensionally-regularized L-loop MI's:

\[
\tilde{I}_{\tilde{V}}(n) = \int \left( \prod_{i=1}^{L} \frac{d^{D-1}k_i}{(2\pi)^D} \right) \tilde{D}^{-\nu_{e1}}_{11} \cdots \tilde{D}^{-\nu_{m}}_{NN} \tilde{D}^{\nu_{11}}_{11} \cdots \tilde{D}^{\nu_{NN}}_{NN} \tag{5}
\]

where the \( \tilde{D} \) = \( (k_i^{2} - m^{2} - \eta) \)

- The original (desired) MI is obtained by

\[
\tilde{I}_{V} = \lim_{\eta \to 0^{-}} \tilde{I}_{\tilde{V}}(n)
\]

- Set up ODE as

\[
\frac{\partial}{\partial n} \tilde{J}(n) = A(n) \tilde{I}(n)
\]

- Obtain MI's @ \( n = 0 \) using BC's at \( n = \infty \)
Generally, can't simply decompose aux. MIs into linear combinations of vacuum integrals near $q^2 = \infty$ b/c there are usually more integration regions (i.e., might involve more than one mass scale).

1. Inequivalent integration regions can be characterized by the size of the loop momentum carried by each branch of the diagram.

   - This is either $O(1/Q)$ – large loop momentum
   - or $O(1)$ – small loop momentum

**Examples:**

- One-loop case: only a single branch with loop momentum $l_i$. So two integration regions contribute: L & S.

- Two-loop case: Generally 3 branches, with loop momenta $l_1, l_2, l_1 + l_2 \rightarrow 5$ contributing regions (LLL, LLS, LSL, SLL, SSS)

=> Obtain BCs near $Q = \infty$ by expanding integrands in each region:

**Propagators:**

- (LL...L) all large regions: \[ \frac{1}{(l+p)^2 - m^2 - Kq} \sim \frac{1}{l^2 - Kq} \]

  \( \Rightarrow \) obtain vacuum integrals

- (SS...S) all small: \[ \frac{1}{(l+p)^2 - m^2 - Kq} \sim \frac{1}{q^2} \rightarrow \text{integrals in a} \]

  \( \Rightarrow \) obtain vacuum integrals

- Mixed: \[ \frac{1}{(l_1 + l_2 + p)^2 - m^2 - Kq} \sim \frac{1}{l_1^2 - Kq} \text{ if } l_1 \neq 0, K \neq 0 \]

  \( \Rightarrow \) L & S contributions are decoupled, so we get factorized integrals.
- BCs are chosen at $\eta = \infty$ b/c of simplicity + introduces.

$\Rightarrow$ General propagator

\[ \frac{1}{(k_L + k_S + p)^2 - m^2 - k\eta} \]

\[ K_L = O(\eta^2) \text{ part of loop momentum} \]

\[ K_S = O(1) \text{ part} \]

\[ p = \text{lin comb. of ext. mom.} \]

\[ K = 0 \text{ or } 1 \]

\[ \text{etc.} \]

\[ \text{etc.} \]

if $k_L \neq 0$ or $k_S \neq 0$, then at $\eta \to \infty$,

\[ \frac{1}{(k_L + k_S + p)^2 - m^2 - k\eta} \sim \frac{1}{k_L^2 - k^2} \]

$\Rightarrow$ This results in either single-mass vacuum FIs or simpler integrals (which can be handled with more AMF).

Using AMF iteratively means only single-mass vacuum FIs are the only additional input besides IBP reduction.

$\Rightarrow$ Note that introducing the new parameter $\eta$ into less propagators means one needs to calculate less MIs. This yields improved AMF in terms of computational complexity. So instead we can consider only the first denominator $\tilde{D}_i = k_i^2 - m_i^2 - \eta$

and all others $\tilde{D}_i = D_i = k_i^2 - m_i^2$ WLOG.
Linear Algebra Approach

- How to determine single-mass vacuum FIs

- Assume that the FIs defined by (5) are single-mass vacuum FIs with \( D_i = k_i^2 + m_i^2 + i0^+ \) being the only massive propagator and \( v_i > 0 \).

1. Define a massless \( p \) integral by ignoring the \( i = 1 \) parts:

\[
\hat{I}_{\vec{v}}^{\perp} (k_i^2) = \left( \frac{L}{\Pi} \delta^D(k_i) \right) \frac{D_{\vec{v}_1} \cdots D_{\vec{v}_N}}{D_{\vec{v}_1} \cdots D_{\vec{v}_N}}
\]

\( \vec{v} = (v_2, \ldots, v_N) \), where \( k_i \) is the "external" momentum.

2. Dimensional counting yields

\[
\hat{I}_{\vec{v}}^{\perp} (k_i^2) = (-k_i^2) \frac{(-1)^D}{2} - \vec{v} \cdot \vec{v}, \quad \hat{I}_{\vec{v}}^{\perp} (-1), \quad v = \sum_{i=1}^{N} v_i
\]

and the original integral can be factorized into two parts and evaluated as

\[
\hat{I}_{\vec{v}} = \int \frac{d^Dk_i}{i\pi^{D/2}} \frac{(-k_i^2)^{(-1-D)/2} - v \cdot \vec{v}}{(k_i^2 - 1 + i0^+)^{\nu_i}} \hat{I}_{\vec{v}}^{\perp} (-1) \quad (6)
\]

\[
\hat{I}_{\vec{v}} = \frac{\Gamma(v - LD/2) \Gamma(LD/2 - v + 1)}{(-1)^v \Gamma(v) \Gamma(D/2)} \hat{I}_{\vec{v}}^{\perp} (-1) \quad (7)
\]

so an \( L \)-loop single-mass vacuum FI \( \hat{I}_{\vec{v}} \)
can be determined by an \((L-1)\)-loop massless \( p \) integral \( \hat{I}_{\vec{v}}^{\perp} (-1) \).

\( \hat{I}_{\vec{v}} \) can be calculated via AMF requiring only single-mass vacuum FIs with \((L-1)\) loops + IBP reduction into, Single-mass vacuum FI with \( L \) loops are determined by those with less than \( L \) loops! This works down until \( L = 1 \)
Thus, all single-mass vacuum FIs, and therefore all FIs, can be determined once linear algebraic relations b/w different FIs are found/provided. (valid for any # of loops L & dimensionality D).

Example:

\[ J = (1,1,1) \]

\[ I_{(1,1)} = \int \left( \frac{1}{2\pi i} \int \frac{d^D k_2}{i\pi D/2} \right) \frac{1}{(k_2^2 - 1) k_2^2 (k_1 + k_2)^2} \]

Using (7):

\[ I_{(1,1)} = \frac{\Gamma(3-D) \Gamma(D-2)}{-\Gamma(1) \Gamma(D-2)} \hat{I}_{(1,1)}(-1) \]

\[ \hat{I}_{(1,1)}(-1) = \int \frac{d^D k_2}{i\pi D/2} \frac{1}{k_2^2 (k_2 + p)^2} \]

where \( p^2 = 1 \)

- Calculate \( \tilde{I}_{(1,1)}(-1) \) via AMF by introducing auxiliary integrals:

\[ \tilde{I}_{(1,0)}(\eta) = \int \frac{d^D k_2}{i\pi D/2} \frac{1}{k_2^2 - \eta} \]

\[ \tilde{I}_{(1,1)}(\eta) = \int \frac{d^D k_2}{i\pi D/2} \frac{1}{(k_2 - \eta)(k^2 + p)^2} \]

- Denote \( \bar{J} = (\tilde{I}_{(1,0)}, \tilde{I}_{(1,1)})^T \), I BP reductions yield

\[ \frac{2}{2\eta} \frac{\partial}{\partial \eta} \tilde{J}(\eta) = \begin{pmatrix} \frac{1-\xi}{\eta} & 0 \\ \frac{1-\xi}{\eta (\mu_1)} & \frac{1-2\xi}{1+\eta} \end{pmatrix} \tilde{J}(\eta) \]
- as $\eta \to \infty$, only the integration region $k_\perp \sim O(1/\eta)$ has a nonzero contribution:

$$\tilde{I}_{(1,0)}(\eta) = \eta^{D/2-1} \int \frac{d^D k_\perp}{(2\pi)^D} \frac{1}{k_\perp^2 - 1}$$

$$= \frac{1}{\eta} \left[ \eta^{D/2-1} (-1)^{D/2} \Gamma(1 - D/2) \right]$$

- These are our BCs at $\eta \to \infty$

$$\tilde{I}_{(1,0)}(\eta) = \frac{1}{\eta}$$

- Solving the ODE system with the above BCs determines $\tilde{I}_{(1,1)}(-1) = \tilde{I}_{(1,0)}(1)$, so we can then obtain $\tilde{I}_{(1,1)}$ from our initial expression.

$\rightarrow$ Paper example of a $S$-loop single-mass vacuum $F-I$ as well.

$\rightarrow$ Since $D$ is arbitrary, the strategy is applicable to general theories. Can also sample different dimensionally around some fixed value (e.g. $\eta \ll e$ with small values of $\epsilon$) to put Laurent expansion w.r.t $\epsilon$ to any desired order (see paper for $e^{-\epsilon} \to Y^k \to e^{\epsilon} + X$ calculation results).
• **Machine Learning Application**

- Instead of using linear algebra to solve for FIs, what about training an ML model to do so?
  - Dr. Gleyzer's work on SYMBA may be a place to apply this.

- The presented approach can reduce complex multiloop FIs to a linear algebra problem.

- SYMBA's goal is to symbolically calculate HEP process squared amplitudes $|M|^2$.

$\Rightarrow$ Instead of doing all the linear algebra for each desired FI one at a time, can we train an ML model to take advantage of the presented ideas/relationships to calculate more complex multiloop FIs?

$\Rightarrow$ Bonus since it may be applicable to arbitrary dimensional theories.