Computing Feynman Integrals via Linear Algebra

References: ① arxiv: 2201.11637 hep-th (Lin. Alg.)

② arxiv: hep-ph/0102033 (Difference eqn. algorithm)

③ arxiv: 1711.09572 (Auxiliary Mass Flow method)

④ arxiv: 2201.03593 (write-up of a course on FIs, very long!)

- Outline:
  - General Feynman Integral approach (identities + MIs)
  - Auxiliary Mass Flow method
  - Linear Algebra approach

- Generalities of Feynman Integrals
  - Higher-loop Feynman Integrals (FI) are complicated and hard to compute
  - There are systematic methods to compute them
    - ↳ Direct methods (sector decomp., Mellin-Barnes representation, loop-tree duality, etc...)
    - ↳ Indirect methods (Difference equations, Diff Eqs.)
      - ↳ Unfortunately, you need B.C.s, so direct integration is still required.

- Generalities

N_K - # of loops $\qquad K_i$ - loop momenta, $i = \{1, \ldots, N_K\}$

N_e - # ext. lines $\qquad p_i$ - ext. momenta, $i = \{1, \ldots, N_p\}$

N_δ - # int. lines

$N_p \equiv N_e - 1$

- Propagator Denom $D_i = q_i^2 + m_i^2$; $\quad q_i$ - i^{th} internal line momentum, lin. comb. of p's and K's

$= q_i^2 + m_i^2 - i0 \qquad m_i$ - " " " mass

- A generic FI in D-dim Euclidean K-space is

$$\int [\delta^D k_1] \dots [\delta^D k_{N_k}] \, V_{\gamma\delta}$$

(we assume dimensional regularization $D = 4 - 2\epsilon$ later)

where $[\delta^D k] = \delta^D k / \pi^{D/2}$

generic integrand

$$\boxed{V_{\gamma\delta} = \frac{\prod_{i=1}^{N_p} \prod_{j=1}^{N_k} (p_i \cdot k_j)^{\delta_{ij1}} \prod_{i=1}^{N_k} \prod_{j=1}^{N_k} (k_i \cdot k_j)^{\delta_{ij2}}}{\prod_{i=1}^{N_d} D_i^{\gamma_i}}} \quad (1)$$

$(\gamma_i \geq 0, \; \delta_{ij\ell} \geq 0)$, $\gamma = \{\gamma_1, \dots, \gamma_{N_d}\}$, $\delta = \{\delta_{ij\ell}\}$

- The numerator of (1) is a product of powers of all possible scalar products involving the loop momenta k.
  ↳ total # of products is $N_{sp} = N_p N_k + N_k(N_k+1)/2$

- We can obtain algebraic and integration-by-parts identities.
  - For each denominator in a generic $V_{\gamma\delta}$, $\exists$ an identity

$$\frac{(p\cdot k)_j}{D_j} = \frac{1}{C_j}\left(1 - \frac{D_j - C_j (p\cdot k)_j}{D_j}\right), \quad j = 1, \dots, N_d \quad (2)$$

where $(p\cdot k)_j \subset D_j$, and C_j is the coefficient of $(p\cdot k)_j$ in D_j. Inner products must be all chosen differently.

Ex: $D_j = (q_j^2 + m_j^2) = ((p_j + k_j)^2 + m_j^2)$

$$= (p^2 + k^2 + \underbrace{2(p\cdot k)_j}_{C_j (p\cdot k)_j} + m_j^2)$$

- Applying these identities in sequence to $V_{\gamma\delta}$ as many times as necessary turns it into a sum of new terms not containing $(p\cdot k)_j$ and D_j simultaneously:

$$\boxed{V'_{ni\alpha\beta} = \frac{\prod_{j=1}^{N_{sp}-n} (p\cdot k \text{ irred})_j^{\beta_j}}{\prod_{j=1}^{n} D_{i_j}^{\alpha_j}}, \quad n \le N_d, \quad \alpha_j, \beta_j \geq 0} \quad (3)$$

⊛ One different set of irred. scalar products corresponds to each combination of denominators

↳ subscript shows dependence on n – # of denominators, their specific combination $i = \{i_1, \dots, i_n\}$, and the exponents $\alpha = \{\alpha_1, \dots, \alpha_n\}$, $\beta = \{\beta_1, \dots, \beta_{N_{sp}-n}\}$.

↳ $(p\cdot k \text{ irred})_j$, $j = 1, \dots, N_{sp}-n$ corresponds to scalar products which can't be simplified further with (2).

- Integrating by parts in D dimensions yields a set of identities:

$$\boxed{\begin{aligned} &\int [d^D k_1] \dots [d^D k_{N_k}] \frac{\partial}{\partial (k_j)_\mu} \left((p_\ell)_\mu V'_{ni\alpha\beta} \right) = 0, \\ &\qquad\qquad j = 1, \dots, N_k, \quad \ell = 1, \dots, N_p \\ &\int [d^D k_1] \dots [d^D k_{N_k}] \frac{\partial}{\partial (k_j)_\mu} \left((k_\ell)_\mu V'_{ni\alpha\beta} \right) = 0, \\ &\qquad\qquad j = 1, \dots, N_k \end{aligned}} \quad (4)$$

↳ each $V'_{ni\alpha\beta}$ yields $N_k(N_p + N_k)$ identities.

↳ $V'_{ni\alpha\beta}$ each contain irreducible scalar products. Calculating derivatives introduces more that can be further reduced via (2).

- Identities will contain linear combinations of two types of integrals:
  - (i) Ones containing all n denominators $\{D_{i_1}, \ldots, D_{i_n}\}$
  - (i) Ones missing a single denominator due to algebraic identities.

  i.e., each identity will be a linear combination of integrals like

$$\int [d^D k_1] \ldots [d^D k_{N_k}]\, V'_{n i \alpha, \beta}$$

  with polynomials of degree 0 or 1 in the # of dimensions D as coefficients.

- Generic sets of these identities form a homogeneous linear system of eqns. with the integrals as the unknowns.
  - ↳ such systems are under-determined
  - ↳ $\exists$ "Master integrals" (MI) whose values can't be determined from the system.

=> Generally, FIs can be grouped into families of FIs:

$$I_{\vec{\nu}} = \int \left(\prod_{i=1}^{L} \frac{d^D k_i}{i\pi^{D/2}} \right) \frac{\overbrace{D_{k+1}^{-\nu_{k+1}} \ldots D_N^{-\nu_N}}^{\text{inner products}}}{\underbrace{D_1^{\nu_1} \ldots D_k^{\nu_k}}_{\text{inv. propagators}}}, \quad \begin{array}{l} \nu_1 \ldots \nu_k \in \mathbb{Z} \\ \nu_{k+1} \ldots \nu_N < 0, \mathbb{Z} \end{array}$$

for various values of $\vec{\nu}$. (arxiv: 1004.4199)

FIs form a finite-dim linear space.

⇒ Any FI in a given family can be decomposed into a linear combination of MIs, which forms a finite basis of the linear space formed by the FIs.

↳ Can use integration by parts (IBP) reduction to find MIs for multiloop FIs.

↳ ref ① uses a method using a finite # of identities using carefully chosen parameters i, n, α, β from a large finite set.

W_j represents an FI of a given family

↳ if $\sum_j c_j W_j = 0$ is an identity, one can re-write some of the integrals W_j in terms of other integrals and substitute into the identity, which becomes $\sum_j c_j' W_j' = 0$. Choose a specific integral W_e and write $W_e = \sum_{j \neq e} c_j'' W_j'$, then one can add the new integral to the system & substitute in W_e.

- It's a complicated algorithm, so I omit it here, but any integral can eventually be written as

$$I = \sum_{i=1}^{L} r_e B_e$$

where r_e are rational f^{ns} of D, masses, scalar products of ext. momenta, and B_e are master integrals

$$B_e = \int [d^D k_1] \dots [d^D k_{N_L}] \frac{\prod_{j=1}^{N_{sp}-n} (p \cdot k \text{ invs})_j^{\beta_{jL}}}{D_{i_1} D_{i_2} \dots D_{i_n}}$$

where the combo of indices i & exponents β depend on the index l; ordering of $B_1 \dots B_L$ depends on order of integrals from algo.

- Since FIs can always be written in terms of MIs using IBP reduction, only MIs need to be studied.
  ↳ FIs containing linear propagators can also be determined by FIs containing only quadratic propagators (see reference).

- Auxiliary Mass Flow (AMF) method

- Its possible to compute MIs systematically using differential equation method - rewriting the derivative of an MI wrt its kinematic variables in terms of a linear combination of other MIs using other reduction methods.
  ↳ Still needs Boundary Conditions. Since ∄ general rules to find good BCs, doing this systematically is difficult.

=> Consider auxiliary family of dimensionally-regularized L-loop MIs:

$$\boxed{\tilde{I}_{\vec{\nu}}(\eta) = \int \left(\prod_{i=1}^{L} \frac{d^D k_i}{i\pi^{D/2}} \right) \frac{\tilde{D}_{K+1}^{-\nu_{K+1}} \cdots D_N^{-\nu_N}}{\tilde{D}_1^{\nu_1} \cdots D_K^{\nu_K}}} \quad (5)$$

where the $\tilde{D}_1 \ldots \tilde{D}_K = (k_i^2 - m^2 - \eta)$

- The original / desired MI is obtained by

$$I_{\vec{\nu}} = \lim_{\eta \to i0^-} \tilde{I}_{\vec{\nu}}(\eta)$$

($\vec{\tilde{J}}$ denotes a vector of MIs of the auxiliary family)

- Set up ODEs as

$$\frac{\partial}{\partial \eta} \vec{\tilde{J}}(\eta) = A(\eta) \vec{\tilde{J}}(\eta);$$

$A(\eta)$ an $n \times n$ matrix with entries rationally depending on η.

=> obtain MIs @ $\eta = 0^-$ using BCs at $\eta = \infty$

✪ Generally, can't simply decompose aux. MIs into linear combinations of vacuum integrals near $\eta = \infty$ b/c there are usually more integration regions (i.e., might involve more than one mass scale).

↳ Inequivalent integration regions can be characterized by the size of the loop momentum carried by each branch of the diagram.

↳ This is either $\mathcal{O}(\sqrt{\eta})$ – Large loop momentum
or $\mathcal{O}(1)$ - Small loop momentum

Examples: One-loop case: only a single branch with loop momentum ℓ_1. So two integration regions contribute: L & S.

Two-loop case: Generally 3 branches, with loop momenta $\ell_1, \ell_2, \ell_1 + \ell_2 \to 5$ contributing regions (LLL), (LLS), (LSL), (SLL), (SSS)

⇒ Obtain BCs near $\eta = \infty$ by expanding integrands in each region:

Propagators:

(LL...L) all large regions: $\dfrac{1}{(\ell + p)^2 - m^2 - K\eta} \sim \dfrac{1}{\ell^2 - K\eta}$

↳ obtain vacuum integrals

(SS...S) all small: $\dfrac{1}{(\ell + p)^2 - m^2 - K\eta} \sim \dfrac{1}{-\eta} \to$ integrals in a subfamily

mixed: $\dfrac{1}{(\ell_L + \ell_S + p)^2 - m^2 - K\eta} \sim \dfrac{1}{\ell_L^2 - K\eta}$ if $\ell_L \neq 0, K \neq 0$

↳ L & S contributions are decoupled, so we get factorized integrals.

- BCs are chosen at $\eta = \infty$ b/c of simplicity + introduces.

↳ General propagator

simple b/c η big makes kinematic variables negligible ($m \gg 0$, $p \to 0$), yielding vacuum integrals with 0-mass

$$\frac{1}{(k_L + k_s + p)^2 - m^2 - \kappa\eta} \; ;$$

k_L – $\mathcal{O}(\sqrt{\eta})$ part of loop momentum

k_s – $\mathcal{O}(1)$ part

p – lin. comb. of ext. mom.

$\kappa = 0$ or 1

etc.

inequivalent integration regions

if $k_L \neq 0$ or $\kappa \neq 0$, then at $\eta \to \infty$,

$$\frac{1}{(k_L + k_s + p)^2 - m^2 - \kappa\eta} \sim \frac{1}{k_L^2 - \kappa\eta}$$

↳ This results in either single-mass vacuum FIs or simpler integrals (which can be handled with more AMF). Using AMF iteratively means only single-mass vacuum FIs are the only additional input besides IBP reduction.

↳ Note that introducing the new parameter η into less propagators means one needs to calculate less MIs. This yields improved AMF in terms of computational complexity. So instead we can consider only the first denominator $\tilde{D}_1 = k_1^2 - m_1^2 - \eta$ and all others $\tilde{D}_i = D_i = k_i^2 - m_i^2$ WLOG.

- Linear Algebra Approach
  - How to determine single-mass vacuum FIs
  - Assume that the $I_{\vec{\nu}}(\eta)$ defined by (5) are single-mass vacuum FIs with $D_1 = k_1^2 + m_1^2 + i0^+$ being the only massive propagator and $\nu_1 > 0$.
    - ↳ Define a massless p integral by ignoring the $i=1$ parts:

$$\hat{I}_{\vec{\nu}'}(k_1^2) = \int \left(\prod_{i=2}^{L} \frac{d^D k_i}{i\pi^{D/2}} \right) \frac{D_{K+1}^{-\nu_{K+1}} \cdots D_N^{-\nu_N}}{D_2^{\nu_2} \cdots D_K^{\nu_K}}$$

$\vec{\nu}' = (\nu_2 \ldots \nu_N)$, where k_1 is the "external" momentum

    - ↳ Dimensional counting yields

$$\hat{I}_{\vec{\nu}'}(k_1^2) = (-k_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1} \hat{I}_{\vec{\nu}'}(-1), \quad \nu = \sum_{i=1}^{N} \nu_i$$

and the original integral can be factorized into two parts and evaluated as

$$I_{\vec{\nu}} = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{(-k_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1}}{(k_1^2 - 1 + i0^+)^{\nu_1}} \hat{I}_{\vec{\nu}'}(-1) \qquad (6)$$

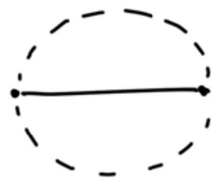
$$I_{\vec{\nu}} = \frac{\Gamma(\nu - LD/2)\Gamma(LD/2 - \nu + \nu_1)}{(-1)^{\nu_1}\Gamma(\nu_1)\Gamma(D/2)} \hat{I}_{\vec{\nu}'}(-1) \quad (7)$$

So an L-loop single-mass vacuum FI $I_{\vec{\nu}}$ can be determined by an (L-1)-loop massless p integral $\hat{I}_{\vec{\nu}'}(-1)$.

☆ Since $\hat{I}_{\vec{\nu}'}$ can be calculated via AMF requiring only single-mass vacuum FIs with (L-1) loops & IBP reduction info, single-mass vaccum FI with L loops are determined by those with less than L loops! This works down until L = 1

⊛ Thus, all single-mass vacuum FIs, and therefore all FIs, can be determined once linear algebraic relations b/w different FIs are found/provided. (valid for any # of loops L & dimensionality D).

Example:



Two-loop single-mass integral. Choosing $m_1^2 = 1$

$\vec{\nu} = (1,1,1)$

$$I_{(1,1,1)} = \int \left(\prod_{i=1}^{2} \frac{d^D k_i}{i\pi^{D/2}} \right) \frac{1}{(k_1^2 - 1)\, k_2^2\, (k_1 + k_2)^2}$$

using (7):

$$I_{(1,1,1)} = \frac{\Gamma(3-D)\,\Gamma(D-2)}{-\Gamma(1)\,\Gamma(D-2)}\, \hat{I}_{(1,1)}(-1)$$

$$\hat{I}_{(1,1)}(-1) = \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{k_2^2 (k_2 + p)^2}$$

where $p^2 = 1$

- Calculate $\hat{I}_{(1,1)}(-1)$ via AMF by introducing auxiliary integrals

$$\left.\begin{aligned} \tilde{I}_{(1,0)}(\eta) &= \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{k_2^2 - \eta} \\ \tilde{I}_{(1,1)}(\eta) &= \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{(k_2^2 - \eta)(k_2^2 + p)^2} \end{aligned}\right\} \text{MIs of the aux. family}$$

- Denoting $\vec{\tilde{J}} = (\tilde{I}_{(1,0)}, \tilde{I}_{(1,1)})^T$, IBP reductions yield

$$\frac{\partial}{\partial \eta} \vec{\tilde{J}}(\eta) = \begin{pmatrix} \frac{1-\varepsilon}{\eta} & 0 \\ \frac{1-\varepsilon}{-\eta(1+\eta)} & \frac{1-2\varepsilon}{1+\eta} \end{pmatrix} \vec{\tilde{J}}(\eta)$$

- as $\eta \to \infty$, only the integration region $|k_2| \sim O(\sqrt{\eta})$ has a nonzero contribution:

$$\tilde{I}_{(1,0)}(\eta) = \eta^{D/2-1} \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{k_2^2 - 1}$$

$$= \eta^{D/2-1} (-1) \Gamma(1 - D/2)$$

$$\tilde{I}_{(1,1)}(\eta) = \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{(k_2^2 - \eta) k_2^2}$$

$$= \tilde{I}_{(1,0)}(\eta) \frac{1}{\eta}$$

These are our BCs at $\eta \to \infty$

- Solving the ODE system with the above BCs determines $\hat{I}_{(1,1)}(-1) = \tilde{I}_{(1,1)}(i0^-)$, so we can then obtain $I_{(1,1,1)}$ from our initial expression.

→ Paper example of a 5-loop single-mass vacuum FI as well

→ Since D is arbitrary, the strategy is applicable to general theories. Can also sample diffrnt dimensionality around some fixed value (e.g. $4-2\varepsilon$ with small values of ε) to fit Laurent expansion wrt ε to any desired order (see paper for $e^+e^- \to \gamma^* \to t\bar{t} + X$ calculation results).

- Machine Learning Application
  - Instead of using linear algebra to solve for FIs, what about training an ML model to do so?
    - Dr. Gleyzer's work on SYMBA may be a place to apply this
    - The presented approach can reduce complex multiloop FIs to a linear algebra problem.
    - SYMBA's goal is to symbolically calculate HEP process squared amplitudes $|M|^2$.

⇒ Instead of doing all the linear algebra for each desired FI one at a time, can we train an ML model to take advantage of the presented ideas/relationships to calculate more complex multiloop FIs?

↳ Bonus since it may be applicable to arbitrary dimensional theories