University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 3: Quantum Mechanics

17 August 2022, 3:00 pm - 6:00 pm

General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 180 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. Consider a particle obeying the Schrödinger equation,

$$i\frac{df(\phi)}{d\phi} = \lambda f(\phi),$$

subjected to the condition,

$$f(\phi) = f(\phi + 2\pi).$$

   (a) (**10 points**) Find the eigenvalues and the normalized eigenfunctions. Here, we consider the range of ϕ to be $0 \le \phi \le 2\pi$.

   (b) (**10 points**) Suppose that the normalized eigenfunction of the particle is $f(\phi) = N\cos^2(\phi)$ with a normalization constant N. What is the probability that a measurement of the eigenvalue yields $n = -2$?

2. (**20 points**) Show that the eigenvectors of any hermitian operator are orthogonal to each other if the eigenvalues are different.

3. Consider a particle of mass m in a 1D infinite square potential well of width L, defined so the potential $V(x) = 0$ for $-L/2 \leq x \leq L/2$, and is ∞ outside.

   (a) (**15 points**) Show that the following wavefunction is a steady-state solution:

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) & -\frac{L}{2} \leq x \leq \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

   (b) (**5 points**) What is the energy of the particle in that state?

4. (**20 points**) Consider a quantum system with two energy eigenstates: $|0\rangle$ with energy $E_0 = 0$ and $|1\rangle$ with energy $E_1 = \epsilon$.

   The system is prepared to be in the following superposition $|\alpha(0)\rangle$:

   $$|\alpha(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

   After a time t, it is observed with an operator O that can be represented as

   $$O = |1\rangle\langle 0| + |0\rangle\langle 1|.$$

   What is the expectation value of $\langle O \rangle (t)$?

5. Consider the Hermitian matrix

$$H(\lambda) = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}.$$

(a) (**10 points**) Find eigenvalues in first order and eigenvectors in zeroth in $|\lambda| \ll 1$ using quantum mechanical perturbation theory.

(b) (**10 points**) Find the exact eigenvectors and the exact eigenvalues of $H(\lambda)$.

6. Consider the three-dimensional "rotator", i.e., a system with only rotational degrees of freedom (an example is a particle confined to the surface of a sphere). The angular momentum is $l = 1$, so that the common eigenstates $|l, m\rangle$ of $\hat{\mathbf{l}}^2$ and $\hat{\mathbf{l}}_z$ can serve as basis states, where $l = 1$ and $m = 0, \pm 1$. Let the rotator be in the state

$$|\psi\rangle = \frac{1}{2}(|1,1\rangle + \sqrt{2}|1,0\rangle + |1,-1\rangle).$$

(a) (**8 points**) Calculate the expectation values of the z-component of the angular momentum $\hat{\mathbf{l}}_z$ and of its square $\hat{\mathbf{l}}^2$ for the state $|\psi\rangle$.

(b) (**4 points**) What is the probability that a measurement of the $z-$component of the angular momentum gives the result 0?

(c) (**8 points**) Calculate the expectation value of the $x-$component of the angular momentum $\hat{\mathbf{l}}_x$.
Hint: Use the operators $\hat{\mathbf{l}}_\pm = \hat{\mathbf{l}}_x \pm i\hat{\mathbf{l}}_y$ *with* $\hat{\mathbf{l}}_+|l,m\rangle = \sqrt{l(l+1) - m(m+1)}\hbar|l, m+1\rangle$ *and* $\hat{\mathbf{l}}_-|l,m\rangle = \sqrt{l(l+1) - m(m-1)}\hbar|l, m-1\rangle$.

University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 4: Thermal Physics

16 August 2022, 3:00 pm - 4:30 pm

General Instructions

- Do any 2 of the 3 questions. Indicate clearly which 2 questions that you wish to have graded. Each question is worth 20 points.
- 90 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. Consider a 2-state quantum system consisting of a ground state $|0\rangle$ with energy $E_0 = 0$ and an excited state $|1\rangle$ with energy $E_1 = \epsilon$, which is in thermal equilibrium at a temperature T. Assume the states have no degeneracies.

   (a) (**3 points**) Write down the partition function, $Z(T)$, as a function of temperature.

   (b) (**8 points**) What is the probability the system will be in the ground state $|0\rangle$? What is the probability it will be in the excited state $|1\rangle$?

   (c) (**4 points**) What is the average energy of the system, $\langle U \rangle$, as a function of temperature?

   (d) (**3 points**) What is the heat capacity of the system, $C_V(T) \equiv \left(\frac{\partial U}{\partial T}\right)_V$?

   (e) (**2 points**) In the limit $T \to \infty$, you should find that $C_V(T) \to 0$ – the average energy stops increasing, no matter how much further you increase the temperature! Explain qualitatively in words why this happens.

2. We consider N classical particles of mass m in a harmonic trap with Hamiltonian

$$H = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} + a\mathbf{r}_i^2 \right),$$

where $a > 0$ is a constant that characterizes the confining potential.

(a) (**10 points**) In the microcanonical ensemble, find the phase space volume $\Phi(E)$, characterized by the condition $H(p,q) \leq E$, and the entropy according to the relation $S(E,N) = k_B \ln \Phi(E,N)$, where k_B is the Boltzmann constant. Further resolve the relation for $E = U(S,N)$, and use the definition of temperature $T = (\partial U/\partial S)_N$ to obtain the relation between U and T (the equation of state).

Hint: Stirling's formula reads $\ln N! = N \ln N - N + \frac{1}{2}\ln(2\pi N)$ *for large integer* N*; the volume of an* n*-dimensional unit sphere is* $C_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$*; in the expression for* S*, you may neglect terms of order* $\ln N$.

(b) (**10 points**) Find the canonical partition function Z, the free energy $F(T,N) = -\frac{1}{\beta}\ln Z$, and the equation of state $U(T,N) = -\partial_\beta \ln Z$, where $\beta = \frac{1}{k_B T}$.

Hint: You may find the following Gaussian integral useful: $\int_{-\infty}^{\infty} dx\ e^{-bx^2} = \sqrt{\frac{\pi}{b}}$*; Stirling's formula is stated in part (a) of this problem; in the expression for* $\ln Z$*, you may neglect terms of order* $\ln N$.

3. Consider a thermalized gas consisting of a large number (N) of point particles with mass (m) inside a rigid, cubic container of volume $V = L^3$. Assume that the particles move randomly, and the average of the squared momentum in each direction over the N particles is identical, $\overline{p_x^2} = \overline{p_y^2} = \overline{p_y^2} = \frac{1}{3}\overline{p^2}$, where $\overline{p^2}$ is the average of the squared momentum.

   (a) (**7 points**) Considering the collision of a particle with the wall of the container (you can consider the motion of a particle only in the $x-$direction), derive

   $$PV = \frac{2}{3}K,$$

   where P is the pressure of the gas, and $K = N \times \frac{1}{2}m\overline{v^2}$ is the total kinetic energy of the gas with the average of the squared speed $\overline{v^2}$.

   (b) (**7 points**) When $\overline{v^2}$ is close to c^2, where c is the speed of light, we need to reconsider the above derivation by extending Newton's theory to Einstein's special theory of relativity. In the Einstein's theory, the average of squared momentum and the average energy of a gas particle are given by

   $$\overline{p^2} = \frac{m^2\overline{v^2}}{1 - \frac{\overline{v^2}}{c^2}}, \quad \overline{E} = \frac{mc^2}{\sqrt{1 - \frac{\overline{v^2}}{c^2}}},$$

   respectively. By using the above formulas, derive

   $$PV = N\frac{1}{3}\frac{c^2\overline{p^2}}{\overline{E}}.$$

   (c) (**3 points**) Show that $P \simeq \frac{1}{3}\rho$ in the relativistic limit, $\overline{v^2} \to c^2$, where ρ is the energy density of the gas, $\rho = N\overline{E}/V$.

   (d) (**3 points**) Re-derive $PV \simeq \frac{2}{3}K$ in the non-relativistic limit, $\overline{v^2}/c^2 \ll 1$.