De Sitter and Anti-de Sitter Spaces

De Sitter and anti-de Sitter spaces are solutions to Einstein equations. De Sitter space is directly relevant for observations in two ways. First, evidence shows the early universe had a period of rapid expansion, the 'inflation', which is well approximated by de Sitter spacetime. Second, ~~nowadays~~ nowadays the cosmological constant accounts for about 68% of the energy density of the universe, and this fraction is growing as the universe continue to expand. This means we are entering a 2nd de Sitter phase.

For Einstein's eq.

$$\cancel{R_{\mu\nu}}\ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

in order to find its solutions, people always assume some certain symmetries. The simplest ones are those with the highest degree of symmetry, called maximally symmetric spaces. They have constant Ricci scalar R and are uniquely determined by its value. The most common is the Minkowski space, with $R=0$. The maximally symmetric space with $R>0$ called de Sitter space (dS), while $R<0$ called Anti-de Sitter space (AdS).

A simple way to understand the n-dimensional ~~P~~de Sitter space (dS_n) is by its embedding in an $(n+1)$ dimensional flat space $R^{n,1}$ with cartesian coordinates $(x^0, x^1, \cdots, x^n)$ and metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2$$

De Sitter space is defined as the hyperboloid of radius $a>0$, the hypersurface with equation

$$X_\mu X^\mu = -(X^0)^2 + (X^1)^2 + \cdots + (X^n)^2 = a^2$$

In the case of dS_4, introducing the coordinates (T, χ, θ, ϕ) with

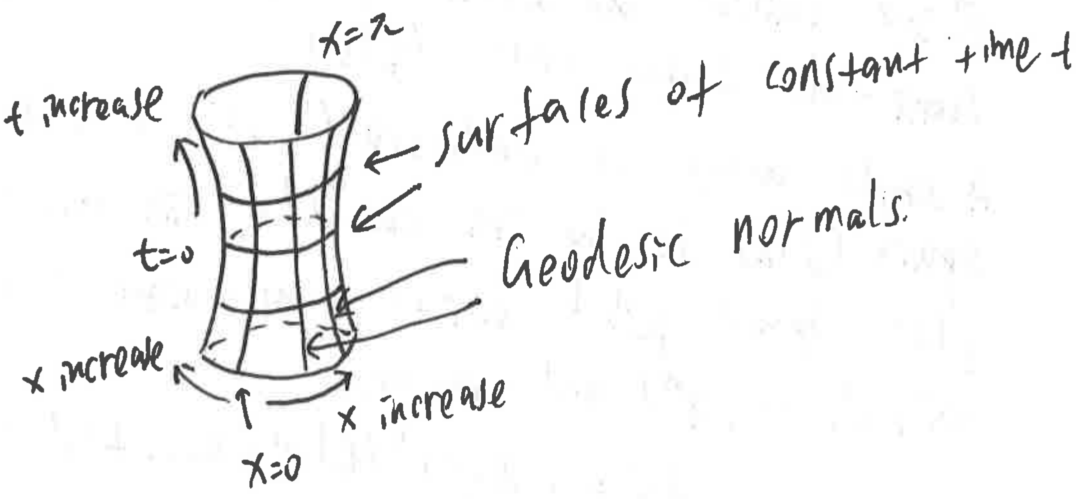
$$X^0 = a\sinh\left(\frac{T}{a}\right), \quad \vec{X} = a\cosh\left(\frac{T}{a}\right)\vec{n}$$

where $\vec{X} = (X^1, X^2, X^3, X^4)$ and $\vec{n} = (\cos\chi, \sin\chi\cos\theta, \sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi)$ with $T\in(-\infty,\infty)$, $0\le\chi\le\pi$, $0\le\theta\le\pi$ and $0\le\phi\le 2\pi$. The the line element is

$$ds^2 = -dT^2 + a^2\cosh^2\left(\frac{T}{a}\right)[d\chi^2 + \sin^2\chi\, d\Omega_2^2].$$

The surfaces of constant time $dT=0$ have metric $dl^2 = d\chi^2 + \sin^2\chi\, d\Omega_2^2$, which represents a 3-dimensional sphere S^3 in spherical coordinates. So, dS_4 is spatially closed.

Disregarding the coordinate singularities at $\chi = 0, \pi$, $\theta = 0, \pi$ and $\phi = 0, 2\pi$, and suppresing two dimensions θ and ϕ, the dS_4 can have a map as:



In this model, the spatial sections contract to a minimum spatial volume (for $T=0$) and then re-expand to infinity. Therefore, the De Sitter space is an expanding space-time for $T>0$.

For the n-dimensional Anti-de Sitter space (AdS_n), it can be embeded in a $(n+1)$-dimensional flat space $R^{n-1,2}$ with metric:

$ds^2 = -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + \cdots + (dX^n)^2$.

So the AdS_n is defined as the hyperboloid with equation:

$$X^\mu X_\mu = -(X^0)^2 - (X^1)^2 + (X^2)^2 + \cdots + (X^n)^2 = -a^2$$

In the case of AdS_4, introducing the coordinates (t, r, θ, ϕ) as:

$$X^0 = a\sin\left(\frac{t}{a}\right)\cosh\left(\frac{r}{a}\right), \quad X^1 = a\cos\left(\frac{t}{a}\right)\cosh\left(\frac{r}{a}\right)$$

$$\vec{X} = a\sinh\left(\frac{r}{a}\right).\vec{n}$$

with $\vec{X} = (X^2, X^3, X^4)$ and $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

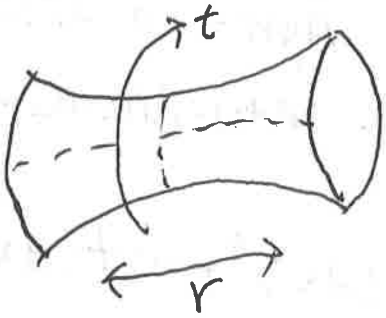
with $0 \le t \le 2\pi a$, $r \ge 0$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

Then the line element is:

$$ds^2 = a^2\left(-\cosh^2\left(\frac{r}{a}\right)dt^2 + dr^2 + \sinh^2\left(\frac{r}{a}\right)d\Omega_2^2\right).$$

In it, the surface of constant time $dt=0$ have metric $dl^2 = dr^2 + \sinh^2\left(\frac{r}{a}\right)d\Omega_2^2$, which represents a hyperbolic space H^3.

So AdS_4 is spatially open.
Disregarding the coordinate singularities at $r=0$, suppressing two dimensions ϕ and θ, a ~~visual~~ visual representation of AdS space is like



Here we have a periodic time coordinate, with points $(0, r, \theta, \phi)$ and $(2\pi a, r, \theta, \phi)$ representing the same point in the space, and this leads to physically unrealistic closed timelike curves. This is physically unrealistic. But since ~~the~~ this metric does not include the time periodicity in its expression, we can unwrap the time coordinate by extending $t \in (-\infty, \infty)$, this is called Anti - de Sitter universal covering space.
These hypersurfaces are called pseudospheres, since they are analogous to a sphere in Euclidean space, $X_\mu x^\mu = a^2$.
The hyperboloid's radius is related to the curvature Ricci scalar as:

$$R_{dS_n} = \frac{n(n-1)}{a^2} \quad , \quad R_{AdS_n} = -\frac{n(n-1)}{a^2}$$

Also, these spaces can be regarded as solutions to the Einstein's equations of empty spaces ($T_{\mu\nu}=0$) with cosmological constant $\Lambda = (n-2)R/2n$.

For dS_4 with

$$ds^2 = -dT^2 + a^2\cosh^2\left(\frac{T}{a}\right)\left[d\chi^2 + \sin^2\chi\, d\Omega_2^2\right]$$

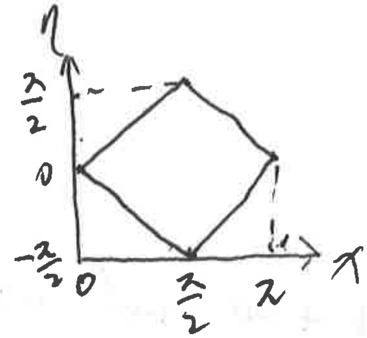
perform the change of variable

$$\cosh\left(\frac{T}{a}\right) = \frac{1}{\cos\eta}$$

$$\Rightarrow ds^2 = \frac{a^2}{\cos^2\eta}\left(-d\eta^2 + d\chi^2 + \sin^2\chi\, d\Omega_2^2\right)$$

with ~~for~~ $-\pi/2 \le \eta \le \pi/2$. $\chi = 0$ and $\chi = \pi$ represents the ~~By using the reduced model~~ antipodal points in the hyperboloid and opposite poles in the pseudosphere.

By using the reduced model with only η and χ coordinates



In it, null geodesics are represented as straight lines with a slope of 45°. Here, given a static observer at $\chi = 0$, a photon sent from a timelike observer in its antipodal point $\chi = \pi$ (that follows a null geodesic) will not reach the first observer (or will reach him for $\eta \to \pi/2$ that is $T \to +\infty$). It means there are pairs of timelike observers in the space that are not causally connected.

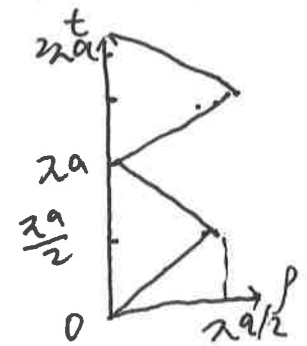
For AdS_4

$$ds^2 = a^2\left(-\cosh^2\left(\tfrac{r}{a}\right) dt^2 + dr^2 + \sinh^2\left(\tfrac{r}{a}\right) d\Omega_2^2\right),$$

set $\sinh(r/a) = \tan(\rho/a)$, there is:

$$ds^2 = \frac{a^2}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho\, d\Omega_2^2\right)$$

for $0 \le \rho \le \pi a/2$. Taking the reduced model with only t and ρ coordinates. Also, due to the unwrapping the time coordinate, the temporal axis extends for $(-\infty, +\infty)$. Here, once the null geodesic reaches the spatial infinity ($\rho = \pi/2$ or $r \to +\infty$), we assume reflecting boundary conditions and the geodesic bounces off and returns. It's like:



In addition, the points at $t=0$ and $t=\pi a$ represent antipodal points while at $t=0$ and $t=2\pi a$ represent the same point in the hyperboloid.

Here, for a static observer at $\rho=0$ a photon emitted from its position will reach spatial infinity $\rho = \pi a/2$ and come back in a finite amount of time.

By performing the change ~~of~~ $\rho = a\sinh(r/a)$ ~~o~~ for the AdS metric,

$$ds^2 = -\left(1+\frac{\rho^2}{a^2}\right)dt^2 + \left(1+\frac{\rho^2}{a^2}\right)^{-1}d\rho^2 + \rho^2 d\Omega_2^2$$

with $\rho \geq 0$. Using this metric and the Euler-Lagrange equations with the Lagrangian $\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$, there ~~TD~~ are:

$$\ddot{t} + \frac{2\rho}{a^2+\rho^2}\dot{\rho}\dot{t} = 0$$

$$\ddot{\rho} + \frac{\rho}{a^2}\left[\left(1+\frac{\rho^2}{a^2}\right)\dot{t}^2 - \frac{1}{\left(1+\frac{\rho^2}{a^2}\right)}\dot{\rho}^2\right] - \rho\left(1+\frac{\rho^2}{a^2}\right)\left[\dot{\theta} + \sin^2\theta\,\dot{\phi}^2\right] = 0$$

$$\ddot{\theta} + \frac{2}{\rho}\dot{\rho}\dot{\theta} - \cos\theta\sin\theta\,\dot{\phi}^2 = 0$$

$$\ddot{\phi} + \frac{2}{\rho}\dot{\rho}\dot{\phi} - 2\frac{1}{\tan\theta}\dot{\theta}\dot{\phi} = 0$$

the dot indicates the derivative with respect to the affine parameter τ. In addition, as the Lagrangian is independent of the time and angular coordinates t and ϕ, $P_t = E$ and $P_\phi = L$ are conserved, that is,

$$\left(1+\frac{\rho^2}{a^2}\right)\dot{t} = E = \text{const}, \quad \text{and}$$

$$\rho^2\dot{\phi} = L = \text{const}.$$

We can set $\theta = \frac{\pi}{2}$ for convenience. (due to the symmetry of the space)

First take a look at circular timelike geodesics when $\rho = \text{const.} \to \dot{\rho} = 0$. Then the geodesic equations are:

$$\ddot{t} = \ddot{\phi} = 0 \quad \text{and}$$

$$\frac{\rho}{a^2} \dot{t}^2 - \rho \dot{\phi}^2 = 0 \Rightarrow \dot{t}/a = \dot{\phi}$$

So the circular geodesics are closed orbits of the form $\phi \propto t/a$ with the same angular velocity $1/a$ regardless of the radial distance. Therefore, we can have a rigidly-rotating frame of reference in AdS.

Setting $\phi' = \phi - t/a = \text{const.}$, $d\phi^2 = dt^2/a^2$, therefore

$$ds^2 = -\left(1 + \frac{\rho^2}{a^2} \cos^2\theta\right) dt^2 \quad (d\rho = d\theta = 0)$$

So, observers at rest in the rotating frame have $ds^2 < 0$ and are timelike.

On the other hand, for the geodesics, $(c = 1)$

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \begin{cases} \pm 1, & \text{timelike/spacelike} \\ 0, & \text{null} \end{cases}$$

Restricting to the radial motion, $L = 0 \Rightarrow \dot{\phi} = 0$, in this case there is

$$-\left(1 + \frac{\rho^2}{a^2}\right) \dot{t}^2 + \frac{\dot{\rho}^2}{\left(1 + \frac{\rho^2}{a^2}\right)} = \begin{cases} \pm 1, & \text{timelike/spacelike} \\ 0, & \text{null} \end{cases}$$

For null radial geodesics, there is

$$\ddot{\rho} = 0$$

For timelike radial geodesics, it is

$$\ddot{\rho} + \frac{1}{a^2}\rho = 0,$$

which is the equation of a simple harmonic oscillator with frequency $\omega = 1/a$. Therefore, radial timelike geodesics will be sinusoidal curves which end up returning to $\rho = 0$ for proper time $\tau = \pi a$ and again at $\tau = 2\pi a$, that have a maximum finite value for $\rho(\tau)$ depending on the initial conditions of $\rho(0)$ and $\rho'(0)$.

For $L \neq 0$, then the constant ~~ϵ~~ ϵ in $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ ~~in~~ includes $\dot{\phi}$ term, ~~the~~ so there is

$$\ddot{\rho} + \frac{1}{a^2}\rho - \frac{L^2}{\rho^3} = 0.$$

By checking numerically, the solutions of this equation are still $2\pi a$-periodic.

So we can see that the null geodesics are linear on proper time and ~~for~~ both radial and circular timelike geodesics are $2\pi a$-periodic.

What we discussed here is part of the basis for some research subjects in progress, such as de Sitter entropy and thermodynamics, understanding of vacuum energy in the context of string theory, etc. of which of the ultimate goal should be the understanding of the cosmological constant problem as consequence of constructed quantum gravity.