Noncommutative space in quantum theory

references : arXiv:hep-th/9912072

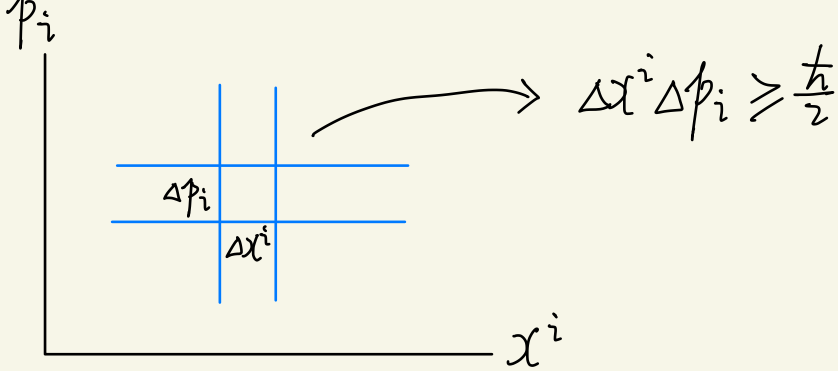
arXiv:hep-th/0007046

In quantum mechanics, the canonical variables $\{x^i, i=1,2,3\}$ with their conjugate momenta $\{p_j, j=1,2,3\}$ are operators satisfying the commutation relations

$$[x^i, p_j] = i\hbar\,\delta^i_j$$

the classical limit is recovered by setting $\hbar \to 0$.

This is the quantization of the phase space.



phase space

What about the quantization of geometry of space itself?

This idea was implemented by Snyder's work in 1947.

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Quantized Space-Time

HARTLAND S. SNYDER
Department of Physics, Northwestern University, Evanston, Illinois

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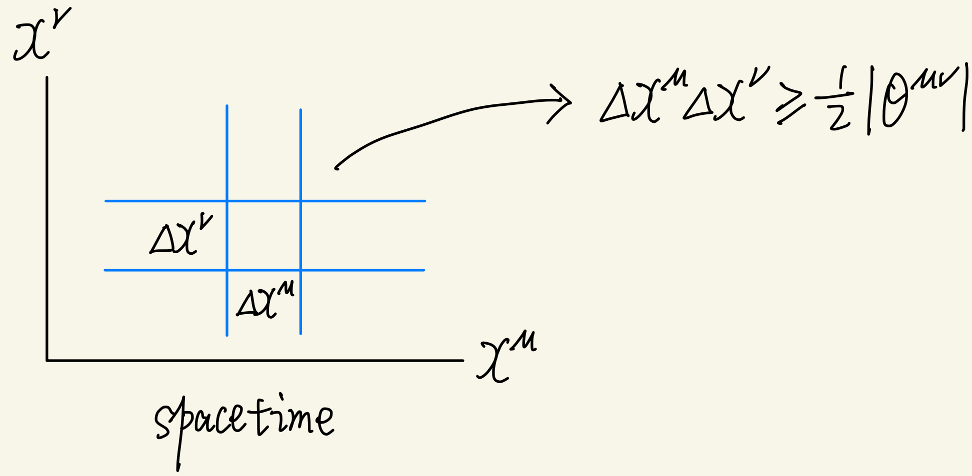
It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time.

The quantization of spacetime $\Rightarrow$ noncommutative spacetime

replacing the spacetime coordinates by the Hermitian operators

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

where $\theta^{\mu\nu}$ is the real antisymmetric matrix.



One of motivations to introduce the noncommutative space is trying to overcome the difficulty of UV divergence in quantum field theory.

$$\text{UV cutoff} \longrightarrow \Lambda_{UV} \sim \theta^{\frac{1}{2}}$$

General relativity at the Planck Length scale l_p?

At the l_p, the quantum effect will play an important role. It is conjuctured that the classical continuum of spacetime structure might be broken down at the Planck length scale.

How to apply to the noncommutativity $\Rightarrow$ Star product

The algebra of functions on the nc-space is realized by replacing the ordinary product among functions by the noncommutative star product

$$(f * g)(x) = \exp\left\{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}\right\} f(x_1)\, g(x_2)\Big|_{x_1=x_2=x}$$

Consider a simple quantum mechanical system consisting of one particle in a potential $V(x,y)$ which is described by the action

$$S = \int dt\, dx^2\, \bar{\psi}\left[i\frac{\partial}{\partial t} - \frac{\vec{p}^{\,2}}{2m} - V(x,y)\right]\psi(t,x,y)$$

The corresponding equation of motion is the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(t,x,y) = \left[\frac{\hat{\vec{p}}^2}{2m} + V(x,y) \right] \psi(t,x,y)$$

Impose the noncommutative conditions on the coordinates

$$[x^i, x^j] = i\theta \epsilon^{ij}, \quad i,j = 1,2$$

with $\epsilon^{12} = 1$.

Under the star product, one gets

$$V(\vec{x}) * \psi(\vec{x}) = V(\vec{x}) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \partial_{i_1} \cdots \partial_{i_n} V(\vec{x}) \, \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \partial_{j_1} \cdots \partial_{j_n} \psi(\vec{x})$$

Since $p_{i_k} = -i \frac{\partial}{\partial x^{i_k}}$, and

$$V(\vec{x}) = \int d^2k \, e^{i\vec{k}\cdot\vec{x}} \, V(\vec{k})$$

introduce the notation $\tilde{p}_{i_k} = \theta^{i_k j_k} p_{j_k}$, one gets

$$\partial_{i_1} \cdots \partial_{i_n} V(\vec{x}) \, \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \partial_{j_1} \cdots \partial_{j_n} \psi(\vec{x})$$
$$= i^n \int d^2x \, e^{i\vec{k}\cdot\vec{x}} \, V(\vec{k}) \, (\vec{k} \cdot \vec{\tilde{p}})^n \psi(\vec{x})$$

Then,

$$V(\vec{x}) * \psi(\vec{x}) = \int d^2k\, e^{i\vec{k}\cdot\vec{x}}\, e^{\frac{i}{2}\vec{k}\cdot\vec{\tilde{p}}}\, V(\vec{k})\, \psi(\vec{x})$$

$$= V(\vec{x} - \tfrac{1}{2}\vec{\tilde{p}})\, \psi(\vec{x})$$

where $\vec{k}\cdot\vec{\tilde{k}} = 0$ and $[x_i, p_j] = i\delta_{ij}$ have been used.

if $$H = \frac{\vec{p}^{\,2}}{2m} + V(x) \longrightarrow$$ interaction in x-direction

$$\Downarrow \text{ noncommutative space}$$

$$H_{nc} = \frac{1}{2m}(p_x^2 + p_y^2) + V(x - \tfrac{1}{2}\theta p_y)$$

$$\psi(x, y) = \phi(x)\, e^{iky}$$

For example, if there is only the harmonic oscillation in the x-direction, and

$$[x, y] = i\theta$$

then

$$H_{nc} = \frac{1}{2m} p_x^2 + \frac{k}{2}(x - \tfrac{1}{2}\theta p_y)^2$$

this is the Hamiltonian of a charged particle in the constant magnetic field in the z-direction with

$$B_z = \frac{2}{q\theta}, \quad m = \frac{4}{k\theta^2}$$

in the Landau gauge

$$\vec{A} = B_z x \hat{y}$$

To see this, expand

$$\begin{aligned}
H_B &= \frac{1}{2m}(\vec{p} - q\vec{A})^2 \\
&= \frac{1}{2m}(p_x^2 + p_y^2) - \frac{q}{m}\vec{p}\cdot\vec{A} + \frac{q^2}{2m}\vec{A}^2 \\
&= \frac{1}{2m}(p_x^2 + p_y^2) - \frac{qB_z}{m}x p_y + \frac{q^2 B_z^2}{2m}x^2
\end{aligned}$$

Next consider a scalar field theory on the noncommutative space. For the quadratic part,

$$\int d^D x\, \phi * \phi = \int d^D x\, \phi^2$$

$$\int d^D x\, \partial\phi * \partial\phi = \int d^D x\, (\partial\phi)^2$$

The interations

$$\mathcal{L}_I = \sum_{n=3} a_n g^{n-2} \phi^n$$

are modified under the star product. The noncommutative interaction vertex of ϕ^n has an extra factor

$$V_{nc}(k_1, \cdots, k_n) = exp\left\{-\frac{i}{2}\sum_{i<j} k_i \tilde{k_j}\right\}$$

where

$$k_i \tilde{k_j} = (k_i)_\mu \theta^{\mu\nu} (k_j)_\nu$$

In the noncommutative field theory, ϕ can be regarded as a matrix. Consider the ϕ^4 theory on the noncommutative Eucliean space $\mathbb{R}^4$ with the action

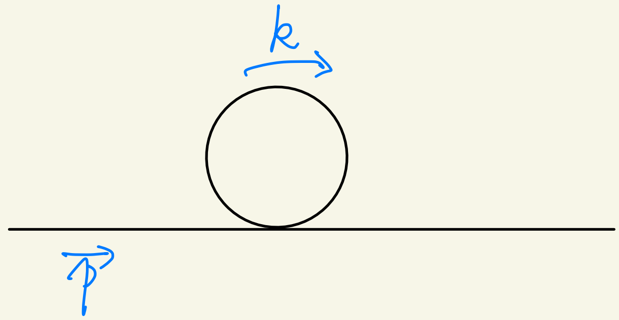
$$S = \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g^2 \phi*\phi*\phi*\phi\right]$$

Denote the 1-particle-irreducible two point function as Γ_{1PI}. At the level,

$$\Gamma_{1PI}^{(0)} = p^2 + m^2$$

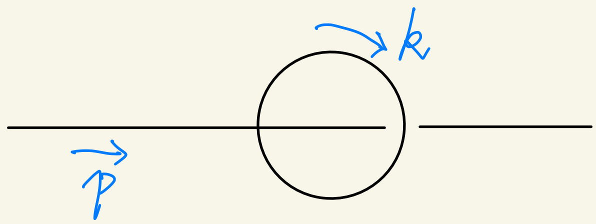
At the 1-loop level, in addition to the planar diagram, there is a nonplanar diagram,

planar



$\Gamma^{(1)\,\mathrm{planar}}_{1PI}$

nonplanar



$\Gamma^{(1)\,\mathrm{nonplanar}}_{1PI}$

When $\theta = 0$, these two diagrams are same (up to a symmetry factor). For $\theta \neq 0$,

$$\Gamma^{(1)\,\mathrm{planar}}_{1PI} = \frac{g^2}{3(2\pi)^4} \int \frac{d^4k}{k^2+m^2}$$

$$\Gamma^{(1)\,\mathrm{nonplanar}}_{1PI} = \frac{g^2}{6(2\pi)^4} \int \frac{d^4k}{k^2+m^2}\, e^{ik\tilde{p}}$$

Use the Schwinger parameter representation of the integral,

$$\frac{1}{k^2+m^2} = \int_0^\infty d\lambda \, e^{-\lambda(k^2+m^2)}$$

then one gets

$$\Gamma_{1PI}^{(1)\,planar} = \frac{g}{48\pi^2}\int \frac{d\lambda}{\lambda^2} e^{-\lambda m^2}$$

$$\Gamma_{1PI}^{(1)\,nonplanar} = \frac{g^2}{96\pi^2}\int \frac{d\lambda}{\lambda^2} e^{-\lambda m^2 - \frac{1}{\lambda} p\circ p}$$

where

$$p\circ q = |p_\mu \theta^{\mu\nu} q_\nu|$$

To regulate $\lambda \to 0$ divergence, introduce the cutoff Λ with $\exp\{-\frac{1}{\Lambda^2\lambda}\}$, then

$$\Gamma_{1PI}^{(1)\,planar} = \frac{g}{48\pi^2}\int \frac{d\lambda}{\lambda^2} e^{-\lambda m^2 - \frac{1}{\Lambda^2\lambda}}$$

$$\Gamma_{1PI}^{(1)\,nonplanar} = \frac{g^2}{96\pi^2}\int \frac{d\lambda}{\lambda^2} e^{-\lambda m^2 - \frac{1}{\lambda}(p\circ p + \frac{1}{\Lambda^2})}$$

and one gets

$$\Gamma_{1PI}^{(1)\,planar} = \frac{g^2}{48\pi^2}\left[\Lambda^2 - m^2 \ln\frac{\Lambda^2}{m^2} + O(1)\right]$$

$$\Gamma_{1PI}^{(1)\,nonplanar} = \frac{g^2}{96\pi^2}\left[\Lambda_{eff}^2 - m^2 \ln\frac{\Lambda_{eff}^2}{m^2} + O(1)\right]$$

with

$$\Lambda_{eff}^2 = \frac{1}{\Lambda^{-2} + p \circ p}$$

The one loop effective action is

$$S_{1PI}^{(1)} = \int d^4p \, \frac{1}{2}\Big\{p^2 + M^2 + \frac{g^2}{96\pi^2 (p \circ p + \Lambda^{-2})}$$

$$- \frac{g^2 M^2}{96\pi^2} \ln\left[\frac{1}{M^2 (p \circ p + \Lambda^{-2})}\right] + \cdots + O(g^4)\Big\} \phi(p)\phi(-p^2)$$

with

$$M^2 = m^2 + \frac{g^2 \Lambda^2}{48\pi^2} - \frac{g^2 m^2}{48\pi^2} \ln\frac{\Lambda^2}{m^2} + \cdots$$

For $p\circ p \ll \Lambda^{-2}$, the effective action in the commutative case is recoverved,

$$S_{1PI}^{(1)\,\text{commutative}} = \int d^4p \, \frac{1}{2}\left(p^2 + M_c^2\right)\phi(p)\phi(-p)$$

For $p\circ p \gg \Lambda^{-2}$ and $\Lambda \to \infty$, $\Lambda_{eff}^2 = (p\circ p)^{-1}$,

$$S_{eff} = \int d^4p \, \frac{1}{2}\left[p^2 + m^2 + \frac{g^2}{96\pi^2 \, p\circ p} - \frac{g^2 m^2}{96\pi^2} \ln \frac{1}{m^2 \, p\circ p} + \cdots + O(g^4)\right]\phi(p)\phi(-p)$$

The UV $\Lambda \to \infty$ limit does not commute with the low momentum IR $p \to 0$ limit, there is a mixing of these two regions called the UV/IR mixing in the noncommutative field theory.