Von Neumann Algebra and Local Quantum Fields

Chuang Xu

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References

✷ arXiv:2112.11614 (Witten 2022)

✷ For more details about von Neumann algebra:
Theory of Operator Algebras I (Takesaki 1979)

Normed space

A **normed space** $\mathscr{N}$ is a linear vector space over field $\mathbb{K}$ (real $\mathbb{R}$ or complex $\mathbb{C}$) on which a norm $\|\cdot\|$ (real-valued function): $\mathscr{N} \to \mathbb{R}$ is defined, satisfying: $\forall x, y \in \mathscr{N}$ and $k \in \mathbb{K}$,

- $\|x\| \geq 0$, equality holds iff $x = 0$.
- $\|kx\| = |k|\|x\|$.
- The triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ holds.

The norm induced metric (distance) is given by $d(x, y) = \|x - y\|$, this gives rise to a locally convex topology (**norm topology**), in which for each $x_0 \in \mathscr{N}$ has an open neighborhood $\mathscr{O}(x_0, \epsilon) = \{x \in \mathscr{N} | d(x, x_0) < \epsilon\}$, where $\epsilon > 0$.

For a **seminorm**, the condition $\|x\| = 0$ iff $x = 0$ is not required.

Banach space and Banach algebra

In a metric space (M, d), a sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ is called **Cauchy sequence** if for any $\epsilon > 0$, $\exists$ a positive integer N such that $\forall i, j > \mathrm{N}$, $d(x_i, x_j) < \epsilon$.

If every Cauchy sequence in a metric space M converges to a limit $\in M$, such a metric space is **complete**.

A complete normed space is called the **Banach space**.

An associative algebra $\mathscr{B}$ over the field $\mathbb{K}$ is called the **Banach algebra** if it is a Banach space and the norm satisfies $\|xy\| \leq \|x\| \, \|y\|$, $\forall x, y \in \mathscr{B}$.

A Banach algebra is said to be **unital** if it contains the multiplication identity.

Hilbert space

A **Hilbert space** $\mathscr{H}$ is: (i) a complex vector space with an inner product $(\ ,\ ) : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ which is sesqui-linear satisfying $\forall \psi, \phi, \xi \in \mathscr{H}$ and $\alpha, \beta \in \mathbb{C}$,

- $(\psi, \phi) = \overline{(\phi, \psi)}$, $(\psi, \alpha\phi + \beta\xi) = \alpha(\psi, \phi) + \beta(\psi, \xi)$.
- $\|\psi\|^2 = (\psi, \psi) \geq 0$, equality holds iff $\psi = 0$.

(ii) a complete metric space in the sense that the norm induced metric is defined from the inner product.

A Hilbert space is a Banach space equipped with the inner product.

If the completion of $\mathscr{H}$ is not required, $\mathscr{H}$ is called the **pre-Hilbert space**.

A linear map $L : \mathscr{H} \to \mathscr{H}$ is called an **operator**. The **adjoint** $L^\dagger$ is defined as $(\psi, L\phi) = (L^\dagger\psi, \phi)$, $\forall \psi, \phi \in \mathscr{H}$.

The operator L on $\mathscr{H}$ is called **bounded** if $\forall \psi \in \mathscr{H}$, $\exists\ \mathrm{M} > 0$ such that $\|L\psi\| \leq \mathrm{M}\|\psi\|$.

An operator is called **unbounded** if it is not bounded.

The **operator norm** of the bounded operator L is defined as

$$\|L\| = \sup_{\psi \neq 0} \frac{\|L\psi\|}{\|\psi\|}, \qquad \forall \psi \in \mathscr{H}.$$

$*$-algebra and C^*-algebra

An **involution** $*$ on a complex Banach algebra $\mathscr{B}$ is a map $* : \mathscr{B} \to \mathscr{B}$ satisfying: $\forall x, y \in \mathscr{B}$ and $\alpha, \beta \in \mathbb{C}$,

- $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$.
- $(xy)^* = y^* x^*$.
- $(x^*)^* = x$.

An algebra with the involution is called $*$-**algebra**.

A C^*-**algebra** is a complex Banach algebra with an involution $*$ satisfying the C^*-identity: $\|x^* x\| = \|x\|^2$, $\forall x \in \mathscr{B}$.

Example

Let $B(\mathscr{H})$ be the collection of bounded operators on a Hilbert space $\mathscr{H}$, then $B(\mathscr{H})$ is a C^*-algebra with the adjoint $\dagger$ as the involution $*$.

Let $\{A_n\}_{n\in\mathbb{Z}_+} \subset B(\mathscr{H})$ be a sequence converging to $A \in B(\mathscr{H})$, $\forall \psi, \phi \in \mathscr{H}$, the topology induced by the convergence is called

norm operator topology	if $\lim\limits_{n\to\infty} \|A_n - A\| = 0$
strong operator topology	if $\lim\limits_{n\to\infty} \|(A_n - A)\psi\| = 0$
weak operator topology	if $\lim\limits_{n\to\infty} (\phi, (A_n - A)\psi) = 0$

weak $<$ strong $<$ norm

Von Neumann algebra

Von Neumann algebra: a unital $*$-subalgebra of $B(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ that is closed in the weak operator topology.

Let N be a von Neumann algebra, its **center** is defined as $Z(N) = N \cap N'$. N is called a **factor** if $Z(N) = \mathbb{C}I$. Let $S \subset B(\mathscr{H})$, the **commutant** of S is defined as $S' = \{x \in B(\mathscr{H}) | \, xy = yx, \forall y \in S\}$.

Bicommutant Theorem

Let $\mathscr{M} \subset B(\mathscr{H})$ be a unital $*$-subalgebra, then the following statements are equivalent:

(i) $\mathscr{M}$ is closed in strong operator topology.

(ii) $\mathscr{M}$ is closed in weak operator topology.

(iii) $\mathscr{M}'' = \mathscr{M}$, where $\mathscr{M}'' = (\mathscr{M}')'$.

Partial isometry and projection

Let $L \in B(\mathscr{H})$, L is called an **isometry** if $(L\phi, L\phi) = (\phi, \phi), \forall \phi \in \mathscr{H}$, i.e., $L^*L = I$. And L is called **unitary** if $L^*L = LL^* = I$.

Denote the **kernel** of L by $\ker L = \{\phi \in \mathscr{H} \,|\, L\phi = 0\}$ and its orthogonal complement by $(\ker L)_\perp$.

L is called a **partial isometry** if L is an isometry on $(\ker L)_\perp$.

L is called a (self-adjoint) **projection** if $L = L^* = L^2$.

Two projections $P_1, P_2 \in B(\mathscr{H})$ are called **orthogonal** if $P_1 P_2 = 0$.

L is called **positive** if $(\phi, L\phi) \geq 0, \forall \phi \in \mathscr{H}$. Denote $L_1 \geq L_2$ if $L_1 - L_2$ is positive.

Classification of von Neumann algebra

The classification of von Neumann algebra is based on its projections.

Let N be a von Neumann algebra. Two projections $P_1, P_2 \in N$ are called **equivalent**, i.e., $P_1 \sim P_2$ if $\exists$ a partial isometry $L \in N$ such that $LL^* = P_1$ and $L^*L = P_2$. Denote $P_1 \preceq P_2$ if $\exists$ a partial isometry $L \in N$ such that $LL^* = P_1$ and $L^*L \leq P_2$.

A projection $P \in N$ is called

finite	if $P \neq 0, Q \leq P \Rightarrow Q = 0$ or $Q = P$
infinite	if P is not finite
purely infinite	if no non-zero finite projection $Q \leq P$
properly infinite	if QP is infinite for every central projection Q

Relative dimension (Murray & von Neumann 1936)

There exists a real-valued function $\mathrm{D}(P)$ defined on the set of projections $\{P_\lambda\} \subset$ factor F with the following properties:

(i) $\mathrm{D}(P) \geq 0$, equality holds iff $\mathrm{P} = 0$.

(ii) $\mathrm{D}(P_1) = \mathrm{D}(P_2)$ if $\exists Q \in F$ such that $P_2 = QP_1Q$.

(iii) $\mathrm{D}(P_1 + P_2) = \mathrm{D}(P_1) + \mathrm{D}(P_2)$ if $P_1P_2 = 0$.

The function $\mathrm{D}(P)$ is called **relative dimension**.

The (normalized) range of the relative dimension function is isomorphic to one of the following sets:

Type $\mathrm{I_n}$	integers $\{0, 1, ..., \mathrm{n}\}$	**Type** $\mathrm{I_\infty}$	integers $\{0, 1, ..., \infty\}$
Type $\mathrm{II_1}$	interval $[0, 1]$	**Type** $\mathrm{II_\infty}$	interval $[0, \infty]$
Type III	numbers $\{0$ and $\infty\}$		

A non-zero projection $P \in N$ is called **minimal** if $Q \leq P \Rightarrow Q = 0$ or $Q = P$, where $Q \in N$. A projection $P \in N$ is called **Abelian** if PNP is Abelian.

Type I factor

A factor is **type** I if it contains a minimal projection.

Type II_1 factor

The type II_1 factor admits a linear map (trace) Tr: $F \to \mathbb{C}$ satisfying that $\forall A, B \in F$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and $\mathrm{Tr}(A^*A) \geq 0$.

Type III factor

A factor is **type** III if it is purely infinite.

Finite quantum mechanical system

For a system with finite (bosonic) degrees of freedom, its canonical variables are a finite set. The quantization is by imposing the commutation relations

$$[x_i, p_j] = i\delta_{ij}, \quad [x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad i, j = 1, ..., n.$$

The Hilbert space $\mathscr{H}$ (irreducible representation of algebra of commutation relations) of this system is *unique* (up to isomorphism), and there is no distinguished vector in $\mathscr{H}$.

But if more structure is attached to $\mathscr{H}$, for example, there is a Hamiltonian operator bounded from below, then the ground state of Hamiltonian is a distinguished vector in $\mathscr{H}$.

Scalar field in a curved background

Assume the background is a D-dimensional manifold (M, g) with "most pluses" signature. Consider a free real scalar field with mass m coupled to the (fixed) background metric g, the action is

$$S[\phi, g] = -\frac{1}{2}\int_M d^D x\sqrt{-\mathrm{g}}(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2)$$

The equation of motion (Klein-Gordon equation) for $\phi(x)$ is

$$(\nabla^\mu\nabla_\mu - m^2)\phi(x) = 0$$

In a general curved spacetime, the existence and uniqueness of solutions to above equation may not hold (Cauchy problem).

Let a spacetime (M, g) be a Lorentzian manifold. A closed submanifold $\Sigma \subset M$ is said to be a **Cauchy (hyper-)surface** if every inextensible timelike curve in M intersects Σ exactly once.

If M admits a Cauchy surface, then M is called **globally hyperbolic**.

For a globally hyperbolic spacetime, M has topology $\mathbb{R} \times \Sigma$, then Klein-Gordon equation has a well-defined initial data consisting of ϕ and its canonical conjugate $\dot{\phi}$ (normal derivative of ϕ) that satisfy the canonical commutation relations

$$[\phi(t, \vec{x}), \dot{\phi}(t, \vec{y})] = i\delta^{(D-1)}(\vec{x} - \vec{y})$$

Unlike the case of finite dimensions, in the infinite dimensions, there is *no uniqueness* of the irreducible representation of algebra of commutation relations among field variables.

Choose a "good" representation that is physically sensible.

For example, in a desired representation, the matrix elements of correlator

$$(\xi, \phi(x_1)\phi(x_2)\cdots\phi(x_n)\chi), \quad \forall \xi, \chi \in \mathscr{H}$$

should be singular at short distance.

Consider the case of a spacetime M with a timelike Killing vector field everywhere. By choosing a suitable coordinate system $\{t, \vec{x}\}$ such that the metric is stationary (independent of t), then there is a Hamiltonian operator (generator of time translations) bounded below.

time-independent spacetime

In this case, the representation of algebra of commutation relations among field variables that admits the above Hamiltonian is *unique* (up to isomorphism).

This representation can be constructed by mode expansion of $\phi(t, \vec{x})$ in terms of positive and negative frequency,

$$\phi(t, \vec{x}) = \sum_p [a_p \tilde{\phi}_p(\vec{x}) e^{-i\omega_p t} + a_p^\dagger \overline{\tilde{\phi}_p}(\vec{x}) e^{i\omega_p t}], \quad \omega_p > 0.$$

where $\sum$ is a discrete sum for closed universe and an integral for open universe.

General situations

QFT in a closed universe $\Rightarrow$ high energy modes (short wavelength) along Σ can be separated to positive and negative frequency and have a natural decomposition in creation and annihilation operators. For low energy modes, no notion of such a decomposition, but since there are only *finitely* many low energy modes, the approximation of asymptotic separation for these modes in creation and annihilation operators is available.

QFT in an open universe (analogous to quantum statistical mechanics at temperature $T = \beta^{-1}$ with infinite volume) $\Rightarrow$ infinitely many modes $\Rightarrow$ no natural choice of a distinguished Hilbert space for quantum filed (there may be many inequivalent constructions)

In quantum mechanics, a system consisting of the subsystems A and B is described by a tensor-product Hilbert space $\mathscr{H}_{\text{system}} = \mathscr{H}_A \otimes \mathscr{H}_B$. A general state of system restricted to the subsystem A is described by a density matrix ρ_A (self-adjoint operator acting on the Hilbert space $\mathscr{H}_A$).

Let $\mathscr{A}_A$ be the algebra of all operators on $\mathscr{H}_A$. For $L \in \mathscr{A}_A$, the expectation value of L in a state described by ρ_A is $F(L) = \text{Tr}(\rho_A L)$. The linear function $F(L)$ is known as a state on an algebra.

In QFT $\Rightarrow$ no way to associate a Hilbert space to a *local region* $\subset$ spacetime.

One can associate to a local region U (open set in spacetime) an algebra of operators $\mathscr{A}_U$.

Difference: the algebra $\mathscr{A}_A$ on the subsystem in quantum mechanics is the von Neumann algebra of Type I, but the algebra $\mathscr{A}_U$ on local region in QFT is the von Neumann algebra of Type III.

The Type III algebra does not have an irreducible representation in a Hilbert space. And the notion of density matrix is also not applicable to the Type III algebra. The useful notion is linear function $F(L)$ that is the state of the algebra $\mathscr{A}_U$.

Linear function $F(L) \Rightarrow$ a local region in QFT

Density matrix $\rho_A \Rightarrow$ a subsystem in quantum mechanics