University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 1: Classical Mechanics

10 January 2022, 1:30 pm - 4:30 pm

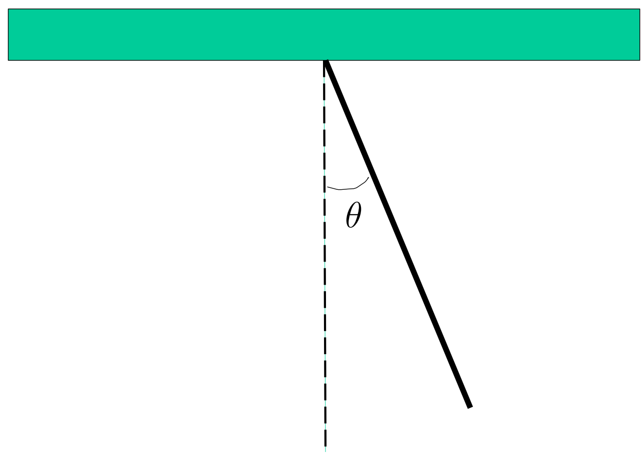
General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 180 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. Consider a pendulum consisting of a thin solid uniform rod of mass m and length L, shown in the figure.

   (a) (**16 points**) Find the Lagrangian and equation of motion for the pendulum. Assume the motion is in a plane.

   (b) (**4 points**) Find the frequency of small oscillations.

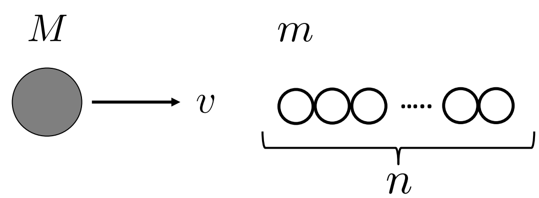


2. n masses, each with a value of $m = 10$ grams, are in contact with each other and form a straight line segment along the $x-$axis. Another mass $M(\neq m)$ moving with speed v along the $x-$axis strikes one of the masses at the end of the line segment. Assume the collision is elastic and that all motion after the collision remains along the $x-$axis.

   (a) (**5 points**) Is it possible for only one mass to be moving after the collision? Explain.

   (b) (**10 points**) If two, and only two, masses are moving after the collision what are their speeds? [Hint: there are two different cases.]

   (c) (**5 points**) If in part (b) the two masses moving after the collision have equal speeds, what are the possible values for M?



3. Consider a circular orbit of a massive object moving within a spherically symmetric gravitational potential produced by a mass distribution with a density described by a function $\rho(r)$. The "rotation curve", i.e. the speed of circular orbits as a function of radius r, is given by a power law of index α, where $-\frac{1}{2} < \alpha \leq 1$:

$$v_c(r) = v_0 \left(\frac{r}{r_0} \right)^{\alpha} .$$

   (a) (**15 points**) What is the form of the density distribution $\rho(r)$, in terms of r, r_0, v_0, and α?

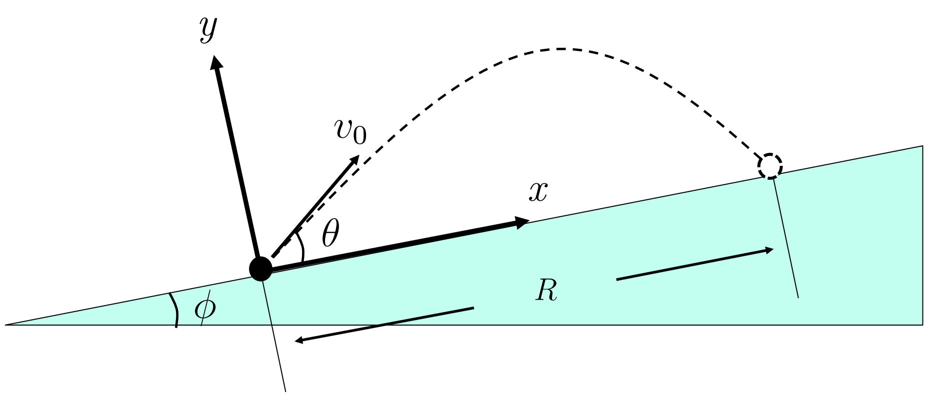
   (b) (**5 points**) No real physical system can fully be described by this rotation curve at all radii. Using the mass contained within a radius r, argue why any real system must deviate from this parametrization at sufficiently large radius.

4. A ball is thrown with initial speed v_0 from an inclined plane. The plane is inclined at an angle ϕ above the horizontal, and the ball's initial velocity is at an angle θ above the plane ($\theta + \phi < 90°$). Choose axes with x measured up the slope and y normal to the slope.

   (a) (**10 points**) Write down Newton's second law using these axes and find the ball's position as a function of time.

   (b) (**5 points**) Show that the ball lands a distance $R = \frac{2v_0^2 \sin\theta \cos(\theta+\phi)}{g\cos^2\phi}$ from its launching point.

   (c) (**5 points**) Show that for a given v_0 and ϕ, the maximum possible range up the incline plane is $R_{max} = \frac{v_0^2}{g(1+\sin\phi)}$.



5. Consider the motion of two particles A and B with masses m_A and m_B, respectively, in the xy-plane. The potential of this system is given by $V(r) = \frac{1}{2}\kappa r^2$, where κ is a positive constant, and r is the distance between the particles.

   (a) (**4 points**) Write the Lagrangian of the system in terms of the position vectors for A and B: $\vec{x}_A = (x_A, y_A)$ and $\vec{x}_B = (x_B, y_B)$, where x_A, y_A, x_B and y_B are Cartesian coordinates.

   (b) (**4 points**) From the Lagrangian, write the Euler-Lagrange equations for the two particles.

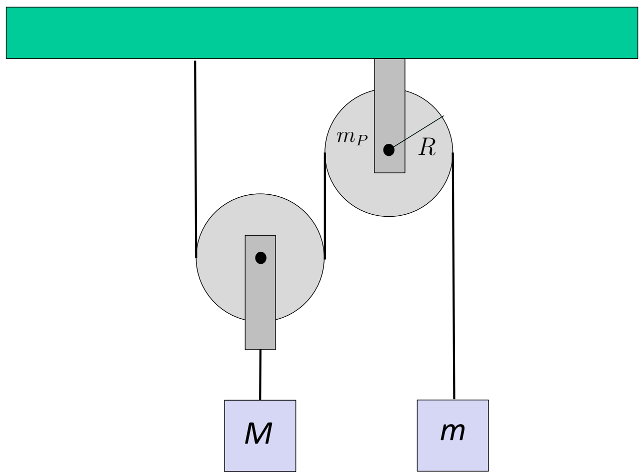
   (c) (**4 points**) Calculate the total momentum of the system, and show that it is conserved.

   (d) (**4 points**) Calculate the total angular momentum of the system, and show that it is conserved.

   (e) (**4 points**) In the center of mass frame, $m_A\vec{x}_A + m_B\vec{x}_B = 0$, express the Lagrangian of the system in terms of $\vec{r} = \vec{x}_A - \vec{x}_B$.

6. In the pulley system shown in the figure, two masses, m and $M > 2m$, are connected with a massless string. The left pulley is massless, while the right one has mass m_P with radius R and its moment of inertia is given by $I = \frac{1}{2}m_P R^2$. Assume that the string does not stretch and the pulleys are frictionless.

   (**20 points**) Express the acceleration of the mass M in terms of m, M, m_P and g (the free-fall acceleration).



University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 2: Electromagnetism

11 January 2022, 1:30 pm - 4:30 pm

General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 180 minutes are allocated for this exam.
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1. Given the vector potential

$$(A_r, A_\theta, A_\phi) = \left(0, 0, g\frac{1-\cos\theta}{r\sin\theta}\right),$$

written in spherical coordinates (r, θ, ϕ),

   (a) (**5 points**) find the resulting magnetic field $\vec{B}$, and the total magnetic flux $\oint \vec{B}\cdot d\vec{S}$ through a sphere of radius r centered at the origin.

   (b) (**5 points**) Does the 'identity' $\nabla \cdot \nabla \times \vec{A} = 0$ hold everywhere, and if not, why not?

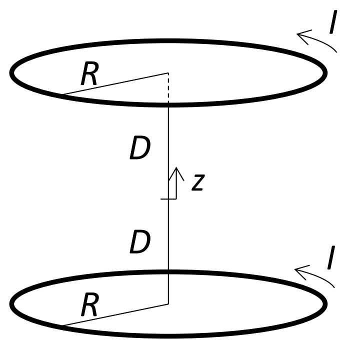
   (c) (**5 points**) Write down the force on an electric charge in this field and show that the angular momentum $\vec{L}$ of the electric charge is not conserved, but its magnitude is.

   (d) (**5 points**) Show $\vec{L} + C\hat{\mathbf{r}}$ is conserved, where $\hat{\mathbf{r}}$ is the unit vector in the radial direction, and C is a constant. What is the value of C?

The curl in spherical coordinates is

$$\nabla \times \vec{A} = \left(\frac{1}{r\sin\theta}\left(\partial_\theta(\sin\theta A_\phi) - \partial_\phi A_\theta\right), \frac{1}{r\sin\theta}\partial_\phi A_r - \frac{1}{r}\partial_r(rA_\phi), \frac{1}{r}\left(\partial_r(rA_\theta) - \partial_\theta A_r\right)\right)$$

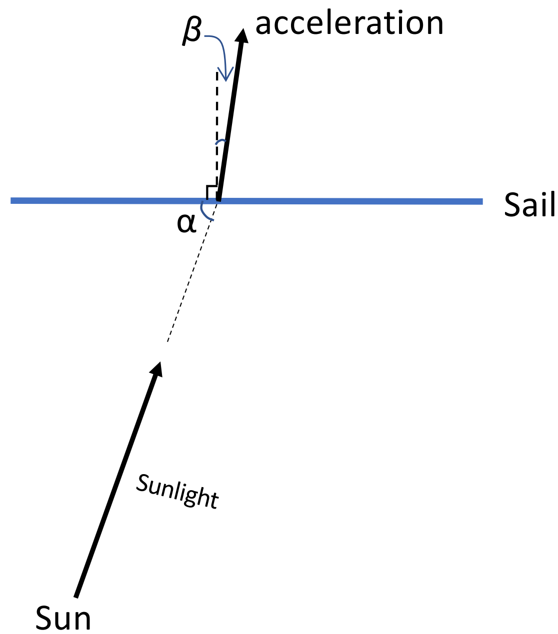
2. Consider two parallel circular wire rings of radius R on a common axis, separated by a distance $2D$. The rings carry the same current I, in the same direction. Defining z as the vertical displacement from the center point, as shown:



(a) (**10 points**) Write down a second order Taylor expansion in the z coordinate around $z = 0$ of the magnetic field $\vec{B}$ along the z axis.

(b) (**10 points**) For what value of D is the field uniform to second order?

3. An idea for interstellar propulsion is a solar sail: a low-mass large-area reflective surface that is propelled by radiation pressure from the Sun. Consider a sail with area A and mass m, that is at an angle α relative to the direction to the Sun. The sail has a reflectivity η, and all photons that are not reflected by the sail are absorbed (make sure you think about both the reflected and absorbed photons).



(a) (**16 points**) If the sail is a distance d from the Sun, which has a total luminosity, i.e. the total amount of energy it emits per unit time, of L_{Sun}, what is the acceleration of the sail due to the momentum imparted by the light? Give both the magnitude of the acceleration, and the angle β that the acceleration vector makes relative to the normal of the sail.

(b) (**4 points**) What does β become in the limit of (i) perfect reflectivity, and (ii) perfect absorption? Give both an explicit expression and an easy-to-understand description of the direction.

4. A long, nonconducting, solid cylinder of radius R has a nonuniform volume charge density ρ that is a function of radial distance r from the cylinder axis: $\rho = b\,r^2$, where b is a constant. Find the electric field for:

   (a) (**10 points**) $r < R$.

   (b) (**10 points**) $r > R$.

5. A hollow cylindrical conductor of outer radius a and inner radius b carries a current i in a direction parallel to the central axis, which is uniformly distributed over its cross-section.

   (a) (**5 points**) Find the magnetic field magnitude B for the radial distance r in the range $b < r < a$.

   (b) (**5 points**) Show that when $r = a$, this equation gives the magnetic field magnitude B at the surface of a long straight wire carrying current i.

   (c) (**5 points**) Show that when $r = b$, it gives zero magnetic field.

   (d) (**5 points**) Show that when $b = 0$, it gives the magnetic field inside a solid conductor of radius a carrying current i.

6. A point charge q and a grounded conducting sphere with radius R are placed as shown in the figure below. In the Cartesian coordinates, the center of the sphere is at the origin and the position of the point charge is $(0, 0, 2R)$.

   (a) (**10 points**) Find the potential at (x, y, z) outside of the sphere.

   (b) (**10 points**) Let us introduce another point charge q' to the system and place it at $(0, 0, -4R)$. Find q' if the force acting on this point charge is zero.