General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.

- 180 minutes are allocated for this exam.

- No reference materials are allowed.

- Do all your work in the corresponding answer booklet (no scratch paper is allowed).

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- No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.
1. Consider a pendulum consisting of a thin solid uniform rod of mass $m$ and length $L$, shown in the figure.

(a) (16 points) Find the Lagrangian and equation of motion for the pendulum. Assume the motion is in a plane.

(b) (4 points) Find the frequency of small oscillations.
2. \( n \) masses, each with a value of \( m = 10 \) grams, are in contact with each other and form a straight line segment along the \( x\)-axis. Another mass \( M(\neq m) \) moving with speed \( v \) along the \( x\)-axis strikes one of the masses at the end of the line segment. Assume the collision is elastic and that all motion after the collision remains along the \( x\)-axis.

(a) (5 points) Is it possible for only one mass to be moving after the collision? Explain.

(b) (10 points) If two, and only two, masses are moving after the collision what are their speeds? [Hint: there are two different cases.]

(c) (5 points) If in part (b) the two masses moving after the collision have equal speeds, what are the possible values for \( M \)?
3. Consider a circular orbit of a massive object moving within a spherically symmetric gravitational potential produced by a mass distribution with a density described by a function $\rho(r)$. The “rotation curve”, i.e. the speed of circular orbits as a function of radius $r$, is given by a power law of index $\alpha$, where $-\frac{1}{2} < \alpha \leq 1$:

$$v_c(r) = v_0 \left( \frac{r}{r_0} \right)^\alpha.$$ 

(a) (15 points) What is the form of the density distribution $\rho(r)$, in terms of $r$, $r_0$, $v_0$, and $\alpha$?

(b) (5 points) No real physical system can fully be described by this rotation curve at all radii. Using the mass contained within a radius $r$, argue why any real system must deviate from this parametrization at sufficiently large radius.
4. A ball is thrown with initial speed $v_0$ from an inclined plane. The plane is inclined at an angle $\phi$ above the horizontal, and the ball’s initial velocity is at an angle $\theta$ above the plane ($\theta + \phi < 90^\circ$). Choose axes with $x$ measured up the slope and $y$ normal to the slope.

(a) (10 points) Write down Newton’s second law using these axes and find the ball’s position as a function of time.

(b) (5 points) Show that the ball lands a distance $R = \frac{2v_0^2 \sin \theta \cos(\theta + \phi)}{g \cos^2 \phi}$ from its launching point.

(c) (5 points) Show that for a given $v_0$ and $\phi$, the maximum possible range up the incline plane is $R_{max} = \frac{v_0^2}{g(1+\sin \phi)}$. 

\[ \text{Diagram:...} \]
5. Consider the motion of two particles A and B with masses $m_A$ and $m_B$, respectively, in the $xy$-plane. The potential of this system is given by $V(r) = \frac{1}{2}\kappa r^2$, where $\kappa$ is a positive constant, and $r$ is the distance between the particles.

(a) (4 points) Write the Lagrangian of the system in terms of the position vectors for A and B: $\vec{x}_A = (x_A, y_A)$ and $\vec{x}_B = (x_B, y_B)$, where $x_A$, $y_A$, $x_B$ and $y_B$ are Cartesian coordinates.

(b) (4 points) From the Lagrangian, write the Euler-Lagrange equations for the two particles.

(c) (4 points) Calculate the total momentum of the system, and show that it is conserved.

(d) (4 points) Calculate the total angular momentum of the system, and show that it is conserved.

(e) (4 points) In the center of mass frame, $m_A\vec{x}_A + m_B\vec{x}_B = 0$, express the Lagrangian of the system in terms of $\vec{r} = \vec{x}_A - \vec{x}_B$. 

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6. In the pulley system shown in the figure, two masses, \( m \) and \( M > 2m \), are connected with a massless string. The left pulley is massless, while the right one has mass \( m_P \) with radius \( R \) and its moment of inertia is given by \( I = \frac{1}{2} m_P R^2 \). Assume that the string does not stretch and the pulleys are frictionless.

**\( 20 \) points** Express the acceleration of the mass \( M \) in terms of \( m, M, m_P \) and \( g \) (the free-fall acceleration).
University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 2: Electromagnetism

11 January 2022, 1:30 pm - 4:30 pm

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1. Given the vector potential
\[ (A_r, A_\theta, A_\phi) = \left( 0, 0, g \frac{1 - \cos \theta}{r \sin \theta} \right), \]
written in spherical coordinates \((r, \theta, \phi)\),

(a) (5 points) find the resulting magnetic field \(\vec{B}\), and the total magnetic flux \(\oint \vec{B} \cdot d\vec{S}\) through a sphere of radius \(r\) centered at the origin.

(b) (5 points) Does the ‘identity’ \(\nabla \cdot \nabla \times \vec{A} = 0\) hold everywhere, and if not, why not?

(c) (5 points) Write down the force on an electric charge in this field and show that the angular momentum \(\vec{L}\) of the electric charge is not conserved, but its magnitude is.

(d) (5 points) Show \(\vec{L} + C \hat{r}\) is conserved, where \(\hat{r}\) is the unit vector in the radial direction, and \(C\) is a constant. What is the value of \(C\)?

The curl in spherical coordinates is
\[
\nabla \times \vec{A} = \left( \frac{1}{r \sin \theta} \left( \partial_\theta (\sin \theta A_\phi) - \partial_\phi A_\theta \right), \frac{1}{r \sin \theta} \partial_\phi A_r - \frac{1}{r} \partial_r (r A_\phi), \frac{1}{r} \left( \partial_r (r A_\theta) - \partial_\theta A_r \right) \right)
\]
2. Consider two parallel circular wire rings of radius $R$ on a common axis, separated by a distance $2D$. The rings carry the same current $I$, in the same direction. Defining $z$ as the vertical displacement from the center point, as shown:

(a) (10 points) Write down a second order Taylor expansion in the $z$ coordinate around $z = 0$ of the magnetic field $\vec{B}$ along the $z$ axis.

(b) (10 points) For what value of $D$ is the field uniform to second order?
3. An idea for interstellar propulsion is a solar sail: a low-mass large-area reflective surface that is propelled by radiation pressure from the Sun. Consider a sail with area $A$ and mass $m$, that is at an angle $\alpha$ relative to the direction to the Sun. The sail has a reflectivity $\eta$, and all photons that are not reflected by the sail are absorbed (make sure you think about both the reflected and absorbed photons).

![Diagram of a solar sail](image)

(a) **(16 points)** If the sail is a distance $d$ from the Sun, which has a total luminosity, i.e. the total amount of energy it emits per unit time, of $L_{\text{Sun}}$, what is the acceleration of the sail due to the momentum imparted by the light? Give both the magnitude of the acceleration, and the angle $\beta$ that the acceleration vector makes relative to the normal of the sail.

(b) **(4 points)** What does $\beta$ become in the limit of (i) perfect reflectivity, and (ii) perfect absorption? Give both an explicit expression and an easy-to-understand description of the direction.
4. A long, nonconducting, solid cylinder of radius $R$ has a nonuniform volume charge density $\rho$ that is a function of radial distance $r$ from the cylinder axis: $\rho = b r^2$, where $b$ is a constant. Find the electric field for:

(a) (10 points) $r < R$.
(b) (10 points) $r > R$. 

5. A hollow cylindrical conductor of outer radius $a$ and inner radius $b$ carries a current $i$ in a direction parallel to the central axis, which is uniformly distributed over its cross-section.

(a) (5 points) Find the magnetic field magnitude $B$ for the radial distance $r$ in the range $b < r < a$.

(b) (5 points) Show that when $r = a$, this equation gives the magnetic field magnitude $B$ at the surface of a long straight wire carrying current $i$.

(c) (5 points) Show that when $r = b$, it gives zero magnetic field.

(d) (5 points) Show that when $b = 0$, it gives the magnetic field inside a solid conductor of radius $a$ carrying current $i$. 
6. A point charge $q$ and a grounded conducting sphere with radius $R$ are placed as shown in the figure below. In the Cartesian coordinates, the center of the sphere is at the origin and the position of the point charge is $(0, 0, 2R)$.

(a) (10 points) Find the potential at $(x, y, z)$ outside of the sphere.

(b) (10 points) Let us introduce another point charge $q'$ to the system and place it at $(0, 0, -4R)$. Find $q'$ if the force acting on this point charge is zero.