University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 3: Quantum Mechanics

18 August 2021, 3:00 pm - 6:00 pm

General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 180 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. In order to completely describe the state of a neutron, *i.e.* both its spin state and its spatial state, we consider the eigenbasis of the spin projection along the z axis, $\hat{\mathbf{S}}_z$, and we represent the neutron state as

$$|\psi(t)\rangle = \begin{pmatrix} \psi_+(\overrightarrow{r}, t) \\ \psi_-(\overrightarrow{r}, t) \end{pmatrix} ,$$

where the respective probabilities of finding the neutron in an infinitesimal volume d^3r around $\overrightarrow{r}$ with its spin component $S_z = \pm\frac{\hbar}{2}$ are

$$d\text{P}(\overrightarrow{r}, S_z = \pm\frac{\hbar}{2}, t) = |\psi_\pm(\overrightarrow{r}, t)|^2 d^3r \ .$$

   (a) (**6 points**) What is, in terms of ψ_+ and ψ_-, the expectation value of the x component of the neutron spin $\langle S_x \rangle$ in the state $|\psi(t)\rangle$?

   (b) (**6 points**) What are the expectation values of the neutron's position $\langle \overrightarrow{r} \rangle$ and momentum $\langle \overrightarrow{p} \rangle$ in the state $|\psi(t)\rangle$?

   (c) (**8 points**) We assume that the state of the neutron can be written:

$$|\psi(t)\rangle = \psi(\overrightarrow{r}, t) \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} ,$$

   where the two complex numbers $\alpha_\pm$are such that $|\alpha_+|^2 + |\alpha_-|^2 = 1$. How do the results of questions (a) and (b) simplify in that case?

   The Pauli matrices are give by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

2. The wave function of a nonrelativistic particle moving in 3 dimensions with mass m is given by

$$\Psi\left(\vec{r},t\right) = \frac{1}{\left(\pi b^2\right)^{\frac{1}{4}}} \exp\left(-\frac{r^2}{2b^2} - i\frac{3\hbar}{2mb^2}t\right),$$

where t is the time, $\vec{r}$ is the position vector, and $r = |\vec{r}|$. Furthermore b is a real valued constant defined by

$$b = \sqrt{\frac{\hbar}{m\omega}}.$$

It has the dimension of a length, and ω is a constant angular frequency.

(a) (**10 points**) What is the energy eigenvalue of the particle?

(b) (**10 points**) Determine the potential energy, and identify the dynamical system.

The Laplace operator in spherical coordinates is given by

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

3. Consider the system with two states defined by the Hamiltonian $H = H_0 + \Delta H$,

$$H_0 = E_0 \begin{pmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}, \qquad \Delta H = \delta\, E_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

where $0 < E_0$, and $0 < \delta \ll 1$ is a small parameter.

(a) (**5 points**) We first consider the limit of $\delta = 0$. Find two eigenvalues of H and corresponding eigenstates in the form of column vector.

(b) (**5 points**) Find the energy eigenvalues at the first order perturbation with respect to $\delta \ll 1$.

(c) (**5 points**) Find the exact eigenvalues of H.

(d) (**5 points**) Expand the exact eigenvalues with respect to δ up to the first order and verify the results found in (b).

4. In a 1-dimensional quantum mechanical system, the wave function of a particle with mass m obeys the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\Psi(t,x) = H\Psi(t,x), \tag{2}$$

where the Hamiltonian of the system is given by

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{3}$$

with a potential $V(x)$. We assume $|\Psi(t,x)|, |\frac{\partial}{\partial t}\Psi(t,x)| \to 0$ for $|x| \to \infty$, faster than $1/\sqrt{|x|}$. Let us define $X(t)$ as the expectation value of the position, $X(t) = \langle x \rangle$.

(a) (**10 points**) Show $\frac{d}{dt}X(t) = \frac{1}{m}\int dx\Psi^*(t,x)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(t,x)$.

(b) (**10 points**) Show that $X(t)$ obeys the equation, m $\frac{d^2}{dt^2}X(t) = -\langle\frac{\partial}{\partial x}V(x)\rangle$, which is a quantum mechanical analog to the Newton's equation of motion.

5. If the x and z components of the angular momentum of a spin-$\frac{3}{2}$ particle are given by the matrices

$$S_x = \frac{\hbar}{2}\begin{pmatrix} & \sqrt{3} & & \\ \sqrt{3} & & 2 & \\ & 2 & & \sqrt{3} \\ & & \sqrt{3} & \end{pmatrix}, \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix}, \tag{4}$$

respectively,

(a) (**5 points**) what is the y component of the angular momentum S_y?

(b) (**5 points**) Compute $S_x^2 + S_y^2 + S_z^2$.

(c) (**4 points**) Say that S_z is measured to be $\frac{\hbar}{2}$. What is the state vector immediately after the measurement?

(d) (**6 points**) What is the probability that S_x is $\frac{\hbar}{2}$ immediately after the measurement?

6. Two identical bosons are in a one-dimensional infinite square well, where the potential is $V(x) = 0$ for $0 \le x \le a$, and $V(x) \to \infty$ elsewhere.

   (a) (**5 points**) Write down the ground state wave function.

   (b) (**5 points**) Write down the ground state wave function for two identical fermions in the same potential.

   (c) (**5 points**) What is the ground state energy for three identical bosons in the potential?

   (d) (**5 points**)What is the ground state energy for three identical fermions in the potential?

University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 4: Thermal Physics

17 August 2021, 3:00 pm - 4:30 pm

General Instructions

- Do any 2 of the 3 questions. Indicate clearly which 2 questions that you wish to have graded. Each question is worth 20 points.
- 90 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. In terms of the number of particles N, temperature T, volume V, the Free Energy of a van der Waals gas is given by:

$$F = -Nk_BT - Nk_BT \ln \frac{n_Q(V - Nb)}{N} - \frac{aN^2}{V}$$

where k_B is the Boltzmann constant, the parameters a and b are positive constants with appropriate dimensions, and the quantum concentration n_Q is equal to:

$$n_Q = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$$

where m is the mass of one atom of the gas.

(a) (**5 points**) Show that the entropy of a van der Waals gas is given by:

$$S = Nk_B \ln \frac{n_Q(V - Nb)}{N} + \frac{5}{2}Nk_B$$

(b) (**5 points**) Derive the expression for the energy of a van der Waals gas in terms of T,N,V and other parameters, and comment briefly on how this compares with the result for a monatomic ideal gas.

(c) (**5 points**) Derive the expression for the pressure of a van der Waals gas in terms of T,N,V and other parameters.

(d) (**5 points**) For temperatures below some critical temperature T_C, the P-V curve for a van der Waals gas along an isotherm has both a local minimum and local maximum. At T_C, however, these disappear and instead the P-V curve has a single point of inflection (that is, neither a maximum nor minimum). Show that this critical temperature T_C and the location in the P-V plane of the inflection point (P_C,V_C) are given by:

$$T_C = \frac{8}{27}\frac{a}{k_Bb},$$
$$P_C = \frac{1}{27}\frac{a}{b^2},$$
$$V_C = 3Nb.$$

2. Say that a certain spin-one nucleus has two possible energy levels ϵ_0 and 0. States with $S_z = \pm\hbar$ have energy ϵ_0, while the $S_z = 0$ state has zero energy. If the nucleus is in thermal contact with a heat bath having temperature T,

   (a) (**7 points**) what is the probability that the nucleus is in the $S_z = \hbar$ state? Take the limit of the result when $T \to 0$ and $T \to \infty$

   (b) (**7 points**) Compute the mean energy and heat capacity for such nuclei in the heat bath, and again take the $T \to 0$ and $T \to \infty$ limits.

   (c) (**6 points**) Compute the mean value of S_z for such nuclei in the heat bath.

3. The entropy for spining particles in a magnetic field B is

$$S(N, E) = C - \frac{kE^2}{2Nm_B^2 B^2} ,$$

where N is the number of particles, E is the energy and C, k, m_B are constants.

(a) (**6 points**) Give an expression for the temperature as a function of N and E. Plot the temperature as a function of energy.

(b) (**7 points**) Say that there are two systems of spinning particles in a common magnetic field. Before coming into thermal contact with each other, system 1) has temperature $T_1 = 100K$, system 2) has temperature $T_2 = 200K$, and system 1) has twice as many particles as system 2). What is the temperature of the combined system after they reach thermal equilibrium?

(c) (**7 points**) Compute the total change in entropy for this process in part (b).