General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.

- 180 minutes are allocated for this exam.

- No reference materials are allowed.

- Do all your work in the corresponding answer booklet (no scratch paper is allowed).

- On the cover of each answer booklet put only your assigned number and the subject - do not write your name.

- Turn in this question sheet with your answer booklet.

- No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.
1. In order to completely describe the state of a neutron, i.e. both its spin state and its spatial state, we consider the eigenbasis of the spin projection along the $z$ axis, $\hat{S}_z$, and we represent the neutron state as

$$ |\psi(t)\rangle = \begin{pmatrix} \psi_+(\vec{r}, t) \\ \psi_-(\vec{r}, t) \end{pmatrix}, $$

where the respective probabilities of finding the neutron in an infinitesimal volume $d^3r$ around $\vec{r}$ with its spin component $S_z = \pm \hbar/2$ are

$$ dP(\vec{r}, S_z = \pm \hbar/2, t) = |\psi_\pm(\vec{r}, t)|^2 d^3r. $$

(a) (6 points) What is, in terms of $\psi_+$ and $\psi_-$, the expectation value of the $x$ component of the neutron spin $\langle S_x \rangle$ in the state $|\psi(t)\rangle$?

(b) (6 points) What are the expectation values of the neutron’s position $\langle \vec{r} \rangle$ and momentum $\langle \vec{p} \rangle$ in the state $|\psi(t)\rangle$?

(c) (8 points) We assume that the state of the neutron can be written:

$$ |\psi(t)\rangle = \psi(\vec{r}, t) \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}, $$

where the two complex numbers $\alpha_\pm$ are such that $|\alpha_+|^2 + |\alpha_-|^2 = 1$. How do the results of questions (a) and (b) simplify in that case?

The Pauli matrices are give by

$$ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$
2. The wave function of a nonrelativistic particle moving in 3 dimensions with mass $m$ is given by

$$\Psi (\vec{r}, t) = \frac{1}{(\pi b^2)^{\frac{3}{4}}} \exp \left( -\frac{r^2}{2b^2} - i\frac{3\hbar}{2mb^2}t \right),$$

where $t$ is the time, $\vec{r}$ is the position vector, and $r = |\vec{r}|$. Furthermore $b$ is a real valued constant defined by

$$b = \sqrt{\frac{\hbar}{m\omega}}.$$

It has the dimension of a length, and $\omega$ is a constant angular frequency.

(a) (10 points) What is the energy eigenvalue of the particle?

(b) (10 points) Determine the potential energy, and identify the dynamical system.

The Laplace operator in spherical coordinates is given by

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$
3. Consider the system with two states defined by the Hamiltonian $H = H_0 + \Delta H$, 

$$H_0 = E_0 \begin{pmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}, \quad \Delta H = \delta E_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $0 < E_0$, and $0 < \delta \ll 1$ is a small parameter.

(a) (5 points) We first consider the limit of $\delta = 0$. Find two eigenvalues of $H$ and corresponding eigenstates in the form of column vector.

(b) (5 points) Find the energy eigenvalues at the first order perturbation with respect to $\delta \ll 1$.

(c) (5 points) Find the exact eigenvalues of $H$.

(d) (5 points) Expand the exact eigenvalues with respect to $\delta$ up to the first order and verify the results found in (b).
4. In a 1-dimensional quantum mechanical system, the wave function of a particle with mass $m$ obeys the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = H \Psi(t, x),$$

(2)

where the Hamiltonian of the system is given by

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

(3)

with a potential $V(x)$. We assume $|\Psi(t, x)|, |\frac{\partial}{\partial t} \Psi(t, x)| \to 0$ for $|x| \to \infty$, faster than $1/\sqrt{|x|}$. Let us define $X(t)$ as the expectation value of the position, $X(t) = \langle x \rangle$.

(a) (10 points) Show $\frac{d}{dt} X(t) = \frac{1}{m} \int dx \Psi^*(t, x) \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi(t, x)$.

(b) (10 points) Show that $X(t)$ obeys the equation, $m \frac{d^2}{dt^2} X(t) = -\langle \frac{\partial}{\partial x} V(x) \rangle$, which is a quantum mechanical analog to the Newton’s equation of motion.
5. If the x and z components of the angular momentum of a spin-$\frac{3}{2}$ particle are given by the matrices

$$S_x = \frac{\hbar}{2} \begin{pmatrix} \sqrt{3} & 2 \sqrt{3} \\ 2 & \sqrt{3} \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 3 & 1 \\ -1 & -3 \end{pmatrix},$$

respectively,

(a) (5 points) what is the y component of the angular momentum $S_y$?

(b) (5 points) Compute $S_x^2 + S_y^2 + S_z^2$.

(c) (4 points) Say that $S_z$ is measured to be $\frac{\hbar}{2}$. What is the state vector immediately after the measurement?

(d) (6 points) What is the probability that $S_x$ is $\frac{\hbar}{2}$ immediately after the measurement?
6. Two identical bosons are in a one-dimensional infinite square well, where the potential is \( V(x) = 0 \) for \( 0 \leq x \leq a \), and \( V(x) \to \infty \) elsewhere.

(a) (5 points) Write down the ground state wave function.

(b) (5 points) Write down the ground state wave function for two identical fermions in the same potential.

(c) (5 points) What is the ground state energy for three identical bosons in the potential?

(d) (5 points) What is the ground state energy for three identical fermions in the potential?
University of Alabama
Department of Physics & Astronomy
Graduate Qualifying Exam
Part 4: Thermal Physics

17 August 2021, 3:00 pm - 4:30 pm

General Instructions

- Do any 2 of the 3 questions. Indicate clearly which 2 questions that you wish to have graded. Each question is worth 20 points.

- 90 minutes are allocated for this exam.

- No reference materials are allowed.

- Do all your work in the corresponding answer booklet (no scratch paper is allowed).

- On the cover of each answer booklet put only your assigned number and the subject - do not write your name.

- Turn in this question sheet with your answer booklet.

- No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.
1. In terms of the number of particles $N$, temperature $T$, volume $V$, the Free Energy of a van der Waals gas is given by:

$$F = -Nk_B T - Nk_B T \ln \frac{n_Q(V - Nb)}{N} - \frac{aN^2}{V}$$

where $k_B$ is the Boltzmann constant, the parameters $a$ and $b$ are positive constants with appropriate dimensions, and the quantum concentration $n_Q$ is equal to:

$$n_Q = \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2}$$

where $m$ is the mass of one atom of the gas.

(a) (5 points) Show that the entropy of a van der Waals gas is given by:

$$S = Nk_B \ln \frac{n_Q(V - Nb)}{N} + \frac{5}{2} Nk_B$$

(b) (5 points) Derive the expression for the energy of a van der Waals gas in terms of $T,N,V$ and other parameters, and comment briefly on how this compares with the result for a monatomic ideal gas.

(c) (5 points) Derive the expression for the pressure of a van der Waals gas in terms of $T,N,V$ and other parameters.

(d) (5 points) For temperatures below some critical temperature $T_C$, the P-V curve for a van der Waals gas along an isotherm has both a local minimum and local maximum. At $T_C$, however, these disappear and instead the P-V curve has a single point of inflection (that is, neither a maximum nor minimum). Show that this critical temperature $T_C$ and the location in the P-V plane of the inflection point $(P_C, V_C)$ are given by:

$$T_C = \frac{8}{27} \frac{a}{K_B b},$$

$$P_C = \frac{1}{27} \frac{a}{b^2},$$

$$V_C = 3Nb.$$
2. Say that a certain spin-one nucleus has two possible energy levels $\epsilon_0$ and 0. States with $S_z = \pm \hbar$ have energy $\epsilon_0$, while the $S_z = 0$ state has zero energy. If the nucleus is in thermal contact with a heat bath having temperature $T$,

(a) (7 points) what is the probability that the nucleus is in the $S_z = \hbar$ state? Take the limit of the result when $T \to 0$ and $T \to \infty$

(b) (7 points) Compute the mean energy and heat capacity for such nuclei in the heat bath, and again take the $T \to 0$ and $T \to \infty$ limits.

(c) (6 points) Compute the mean value of $S_z$ for such nuclei in the heat bath.
3. The entropy for spinning particles in a magnetic field $B$ is

$$S(N, E) = C - \frac{kE^2}{2Nm_B^2B^2},$$

where $N$ is the number of particles, $E$ is the energy and $C, k, m_B$ are constants.

(a) (6 points) Give an expression for the temperature as a function of $N$ and $E$. Plot the temperature as a function of energy.

(b) (7 points) Say that there are two systems of spinning particles in a common magnetic field. Before coming into thermal contact with each other, system 1) has temperature $T_1 = 100K$, system 2) has temperature $T_2 = 200K$, and system 1) has twice as many particles as system 2). What is the temperature of the combined system after they reach thermal equilibrium?

(c) (7 points) Compute the total change in entropy for this process in part (b).