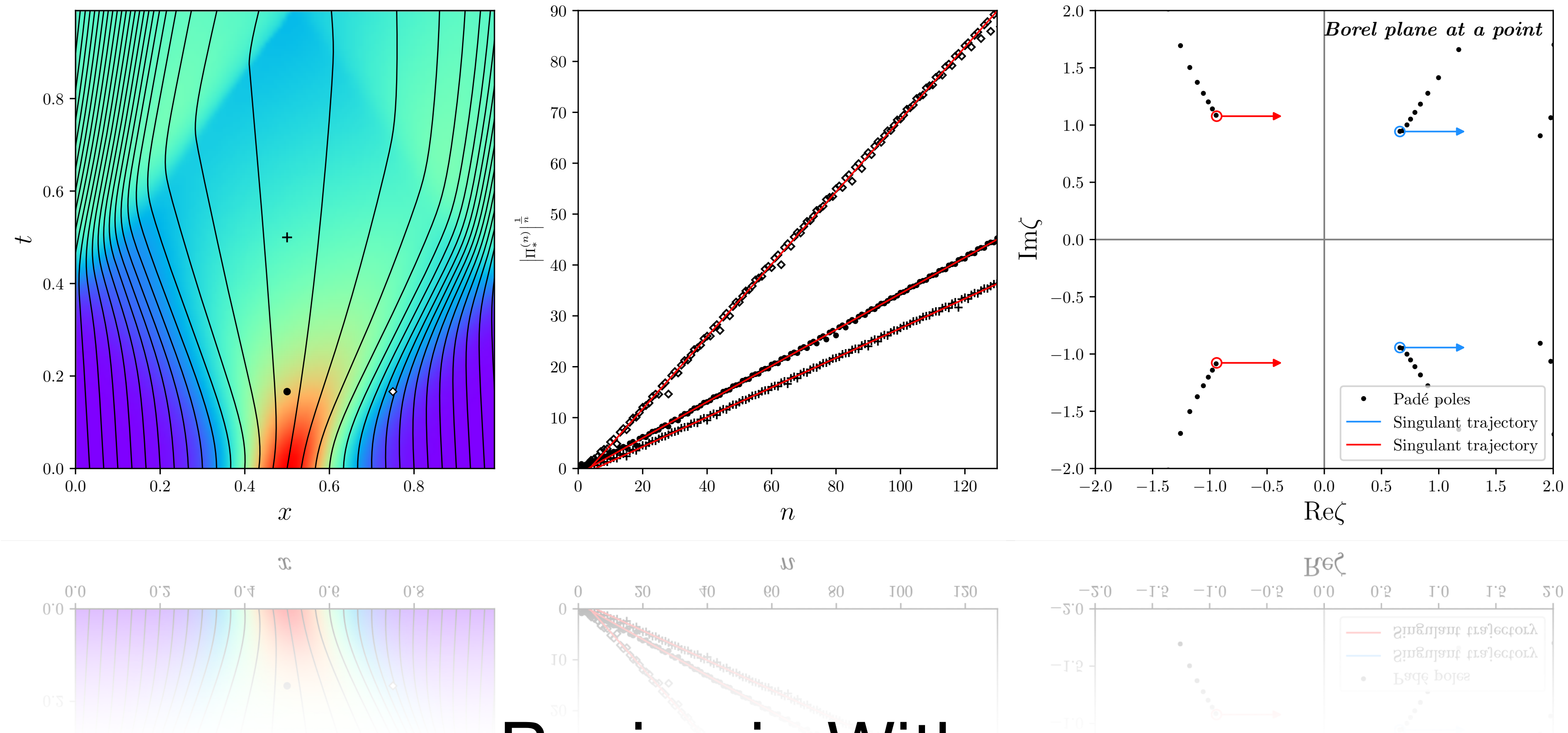
# The large-order hydrodynamic series: quasinormal modes and singulants





Benjamin Withers

University of Southampton


University of Alabama, 12 November 2021

THE ROYAL SOCIETY

STAG RESEARCH CENTRE

Based on:

**1803.08058** - BW

**2007.05524, 2012.15393, 2110.07621** -

M. Heller, A. Serantes, M. Spaliński, V. Svensson, BW

## Motivations

- Hydrodynamics describes conserved currents near equilibrium
- Here: relativistic / conformal hydrodynamics

- **Universal applicability** e.g.
  - astrophysics
  - nuclear physics
  - condensed matter
  - QFTs with holographic duals
    - black hole physics in AdS

- Understand the formal properties of hydrodynamic expansions

## Relativistic hydrodynamics

- Conservation equations $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$

- Hydrodynamic variables $T, U^\mu$

- Constitutive relations $\langle T^{\mu\nu} \rangle = T^{\mu\nu}_{\text{ideal}} + \Pi^{\mu\nu}$

$$\Pi^{\mu\nu} = \sum_{n=1}^{\infty} \Pi^{\mu\nu}_{(n)} [T, U]$$

  - Perturbative expansion in derivatives of hydro variables
  - Symmetries dictate allowed tensor structures $\Pi^{\mu\nu}_{(n)}$
  - Transport coefficients dictated by microscopic details

- Captures non-equilibrium processes in QFTs, black holes, etc.

**Relativistic hydrodynamics**

e.o.s.

$$T^{\mu\nu}_{\text{ideal}} = (\epsilon + P)\, U^\mu U^\nu + P g^{\mu\nu}$$

$$\Pi^{\mu\nu} = -\eta\,\sigma^{\mu\nu} + \tau_\pi \eta\, \mathcal{D}\sigma^{\mu\nu} - \frac{1}{2}\theta_1\, \mathcal{D}_\alpha \mathcal{D}^\alpha \sigma^{\mu\nu} - \theta_2\, \mathcal{D}^{\langle\mu}\mathcal{D}^{\nu\rangle}\mathcal{D}_\alpha U^\alpha + \ldots\,,$$

(some terms suppressed)

shear tensor, one derivative of U

- Transport coefficients $\eta, \tau_\pi, \theta_1, \theta_2, \ldots$ fixed by microscopic details
- e.g. QFT with an Einstein dual $\eta = \frac{1}{4\pi} s$ [Kovtun, Son, Starinets (2004)]

## Relativistic hydrodynamics

$$\Pi^{\mu\nu} = \sum_{n=1}^{\infty} \Pi^{\mu\nu}_{(n)} [T, U]$$

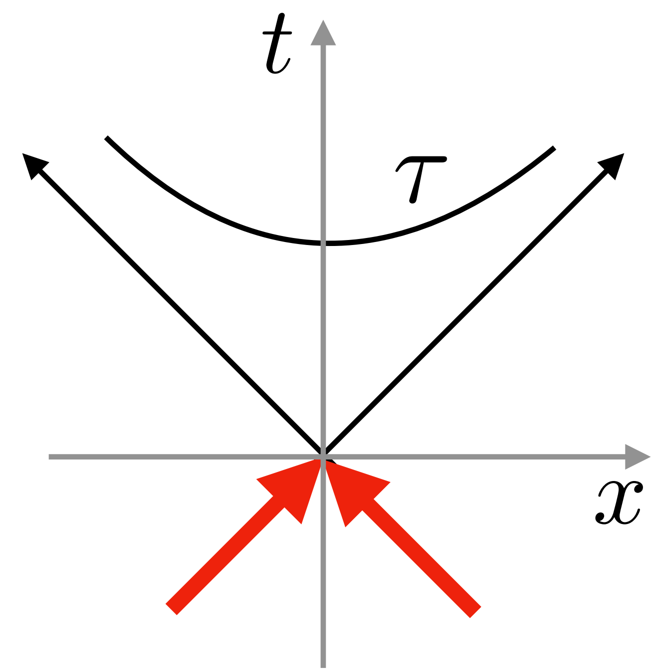
- What is the nature of this series?
- Usually see: “hydro breaks down when gradients become large”
- Goal is to make such statements more precise
  - **If convergent**, what sets the radius?
  - **If divergent**, what is the optimal order of truncation?
- As a classical theory

**Will argue that:**

- It is a divergent series under generic conditions
- Large-order behaviour governed by ‘**singulants**’, which resemble QNMs

## Precursor: **Bjorken flow in holography**

[Heller, Janik, Witaszczyk (2013)]




- Of interest in heavy-ion collisions
- Depends only on $\tau = \sqrt{t^2 - x^2}$
- Large $\tau$ expansion ~ hydro gradient expansion

$$\epsilon = \frac{1}{\tau^{4/3}}\left(\epsilon_2 + \frac{\epsilon_3}{\tau^{2/3}} + \frac{\epsilon_4}{\tau^{4/3}} + \ldots\right)$$

- $\epsilon_n$ are transport coefficients, generated to order 240
- Found to be a **divergent series** $\epsilon_n \sim n!$
- Resummation via Borel-Padé finds non-perturbative contributions

$$\delta\epsilon \sim \tau^{\alpha} \exp\left(-i\tfrac{3}{2}\omega_1(0)\tau^{2/3}\right)$$

- expected since QNMs are non-perturbative in $1/\tau$

$$e^{-\gamma T(\tau)\tau} \qquad T(\tau) \sim \tau^{-1/3}$$

**However, Bjorken is just one flow (& highly symmetric)**

An update on what we have understood since Bjorken flow
In chronological order:

**Part 1. Quasinormal modes**

**1803.08058** - BW

**Part 2. On-shell constitutive relations**

**2012.15393** - M. Heller, A. Serantes, M. Spaliński, V. Svensson, BW

**2110.07621** - M. Heller, A. Serantes, M. Spaliński, V. Svensson, BW

# Part 1. Quasinormal modes

- Quasinormal modes are linear fluctuations around equilibrium
- **In hydrodynamics**
  - aka hydrodynamic modes, fluctuate the hydrodynamic variables

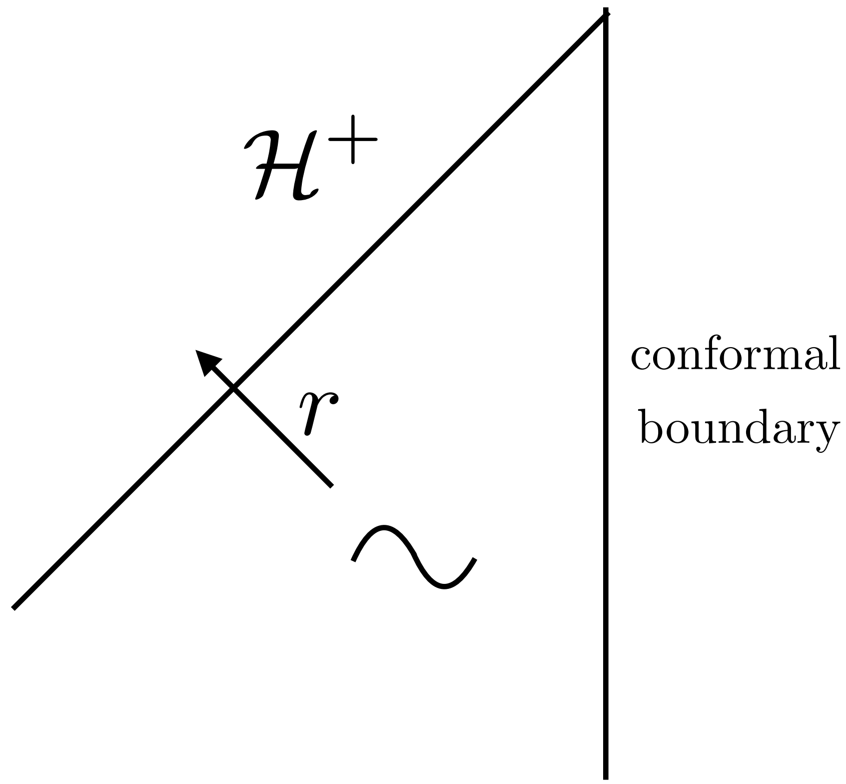
$$T(t,x) = T_0 + \delta T e^{ik\cdot x - i\omega t}$$

$$U(t,x) = \partial_t + \delta U e^{ik\cdot x - i\omega t}$$

  - Solutions: sound and shear
  - e.g. shear $$\omega(k) = -iDk^2 + O(k)^4$$
  - part of a the hydro series $$\omega(k) = \sum_{n=1}^{\infty} \omega_n k^{2n}$$
- **Microscopic theories** (with a hydrodynamic limit)
  - Include sound, shear
  - But have additional non-hydrodynamic modes
  - e.g. black brane in AdS

## QNMs in holographic QFTs

- Holographic $\mathrm{CFT_d}$ $\leftrightarrow$ Einstein gravity in $\mathrm{AdS_{d+1}}$
- Thermal state $\leftrightarrow$ black hole in AdS
- QNMs are given by perturbations of a black hole spacetime




$$\delta g_{\mu\nu} \sim \phi(r) e^{ik\cdot x - i\omega t}$$

e.g. [Kovtun, Starinets (2005)]

- ODE BVP problem $\omega_i(k)$
- Infinitely many of them
- Classify at small $k$:
  - **hydrodynamic** sound and shear
  - **nonhydrodynamic**/transient

$$\omega_i(k \to 0) = 0$$

$$\omega_j(k \to 0) = -\frac{i}{\tau_j}$$

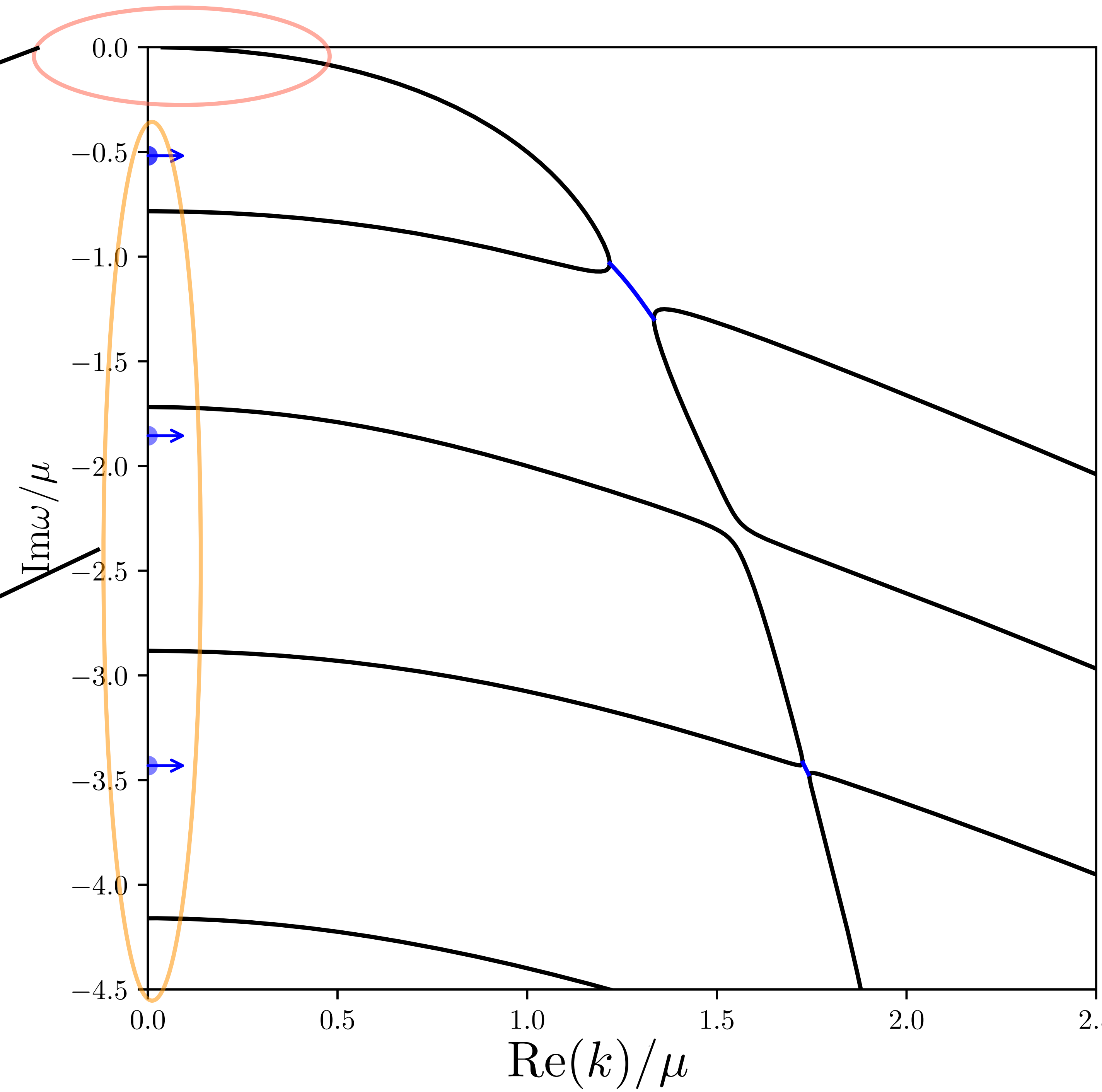
**QNMs of RN-AdS$_4$**

**hydrodynamic**

$$\omega(k) = \sum_{n=1}^{\infty} \omega_n k^{2n}$$

**non-hydro/transients**

$$\omega_j(k) = -\frac{i}{\tau_j} + O(k)^2$$

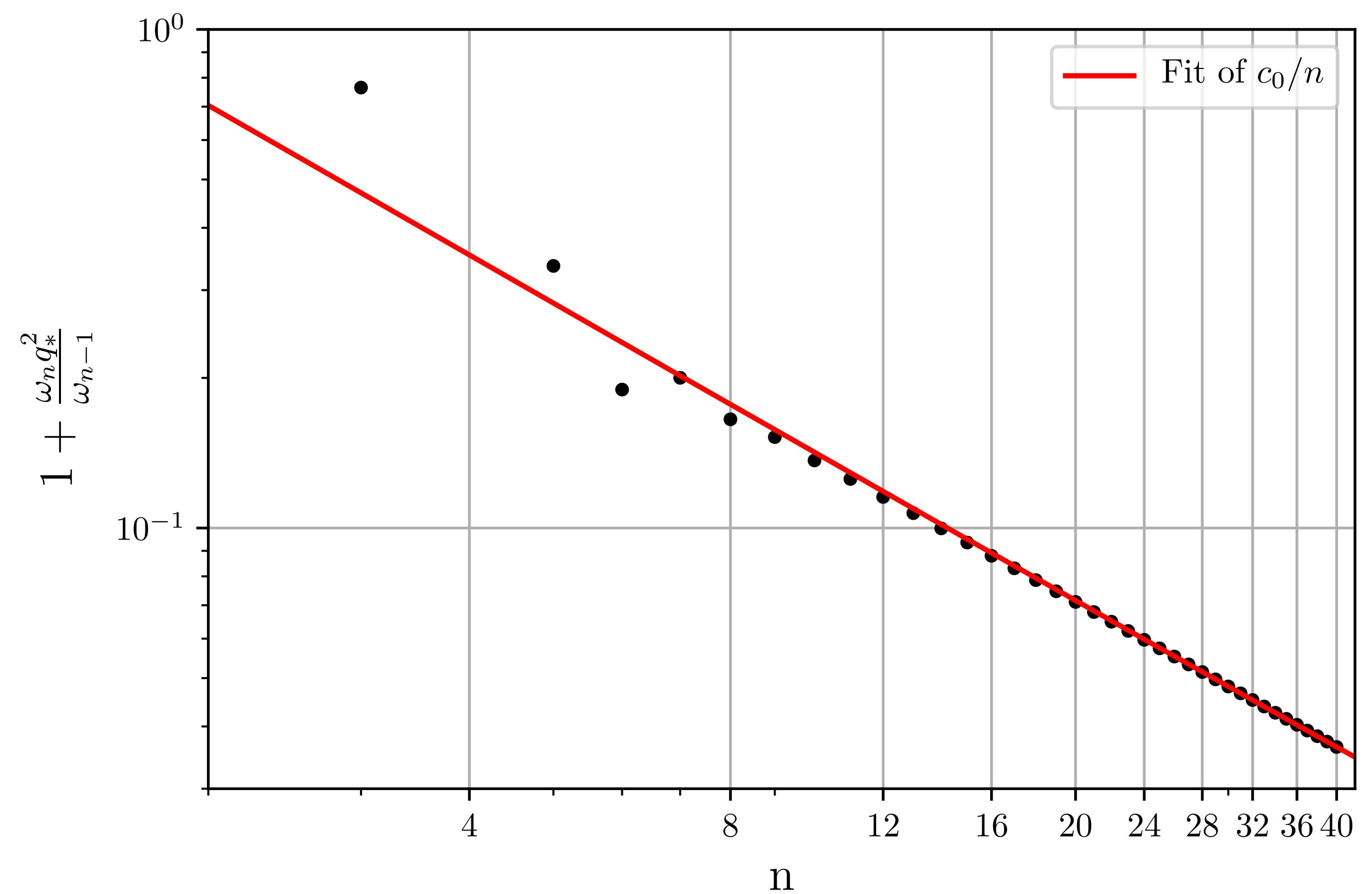



Computed $\omega_n$ to hydrodynamic order ~80 [1803.08058 - BW]

**Radius of convergence**
[1803.08058 - BW]

$$\omega(k) = \sum_{n=1}^{\infty} \omega_n k^{2n}$$

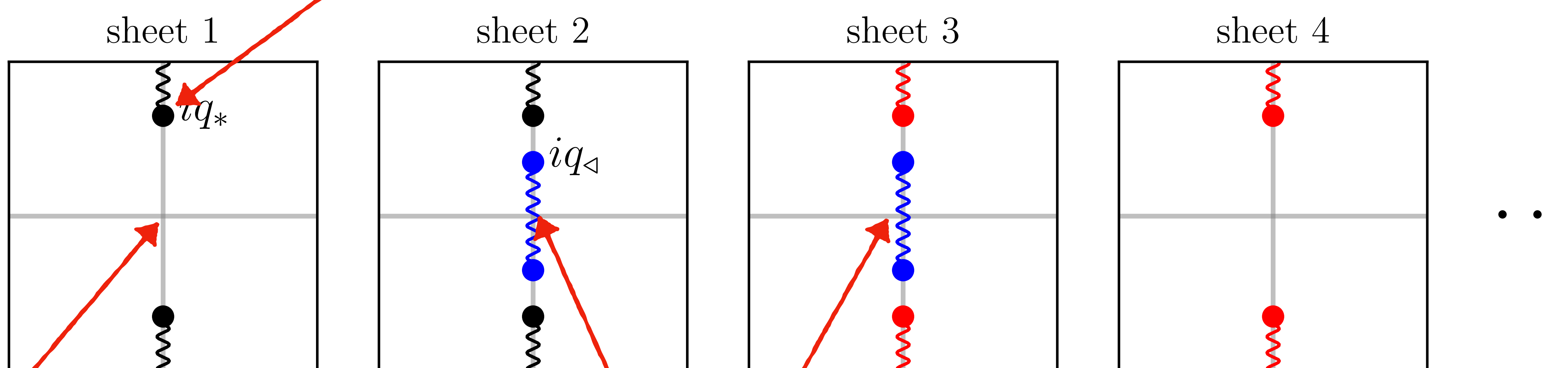
$$r_n \equiv \frac{\omega_n q_*^2}{\omega_{n-1}}$$




- radius of convergence is $q_* \equiv \frac{\epsilon+p}{2\mu\sqrt{\eta}}$
- obstruction are branch point singularities at $k = \pm i q_*$
- Can be revealed by:
  - Using Padé approximant of series data
  - Exact numerical calculations
- View branches of $\omega(k)$ as describing a multi-sheeted Riemann surface

**Complex k plane**

radius of convergence
set by square-root branch point




Hydrodynamics:
neighbourhood of
origin on sheet 1

$$\omega(k) = \sum_{n=1}^{\infty} \omega_n k^{2n}$$

(conjugate pair of)
transient QNMs

$$\omega_j(k) = -\frac{i}{\tau_j} + O(k)^2$$

and more…
infinitely many sheets,
all connected

**analytic continuation**: transport coefficients → black hole QNM spectrum

## Summary of part 1

- branches of $\omega(k)$ for $k \in \mathbb{C}$ describe a Riemann surface
- radius $= |k|$ of closest singularity to origin on hydrodynamic sheet
- Set by mode collisions (*i.e. branch point*) for RN AdS$_4$ [1803.08058 - BW]
- Subsequently observed for several other examples, e.g.
  - Analytically known dispersion relations in MIS/BRSSS
  - RN AdS all Q [Abbasi, Tahery (2020)] [Jansen, Pantelidou (2020)]
  - Near extremal examples [Arean, Davison, Gouteraux, Suzuki (2020)]
  - QNMs with various operator dimensions [Abbasi, Kaminski (2020)]
- **n.b.** 'critical points of spectral curves' [Grozdanov, Kovtun, Starinets, Tadić (2019)] do not determine the radius of convergence
- (counterexamples are provided in: [2012.15393]
  singularities of $\omega(k)$are not always critical points,
  critical points are not always singularities of $\omega(k)$ )

# Part 2. On-shell constitutive relations

$$\langle T^{\mu\nu} \rangle = T^{\mu\nu}_{\text{ideal}} + \Pi^{\mu\nu}$$

$$\Pi^{\mu\nu} = \sum_{n=1}^{\infty} \Pi^{\mu\nu}_{(n)} [T, U]$$

- Wish to evaluate this on-shell
- **Linear response:** intimate connection to QMN in complete generality

Given **initial data** compactly supported in momentum space $k_{\text{max}}$

$$\begin{cases} k_{\text{max}} < k_* & \text{convergent} \\ k_{\text{max}} > k_* & \text{divergent (geometric growth)} \\ k_{\text{max}} \to \infty & \text{divergent (factorial growth)} \end{cases}$$

where $k_*$ is radius of convergence of $\omega(k)$

[2007.05524 - M. Heller, A. Serantes, M. Spaliński, V. Svensson, BW]

**What about nonlinearities?** [Heller, Serantes, Spaliński, Svensson, BW (2021)]

- Easily demonstrated in **BRSSS model**

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0$$
$$\langle T^{\mu\nu} \rangle = T^{\mu\nu}_{\mathrm{ideal}} + \Pi^{\mu\nu}$$
$$\left(1 + \tau_\Pi \mathcal{D}\right) \Pi^{\mu\nu} = -\eta \sigma^{\mu\nu}$$

- n.b. it is not a hydrodynamic theory in the sense we have defined it (not formulated as gradient expansion, includes additional degrees of freedom)

- motivated as a toy model with a good IVP (stability, causality, well-posed)

- It admits a hydrodynamic expansion

$$\Pi^{\mu\nu} = \sum_{n=1}^{\infty} \Pi^{\mu\nu}_{(n)} [T, U]$$

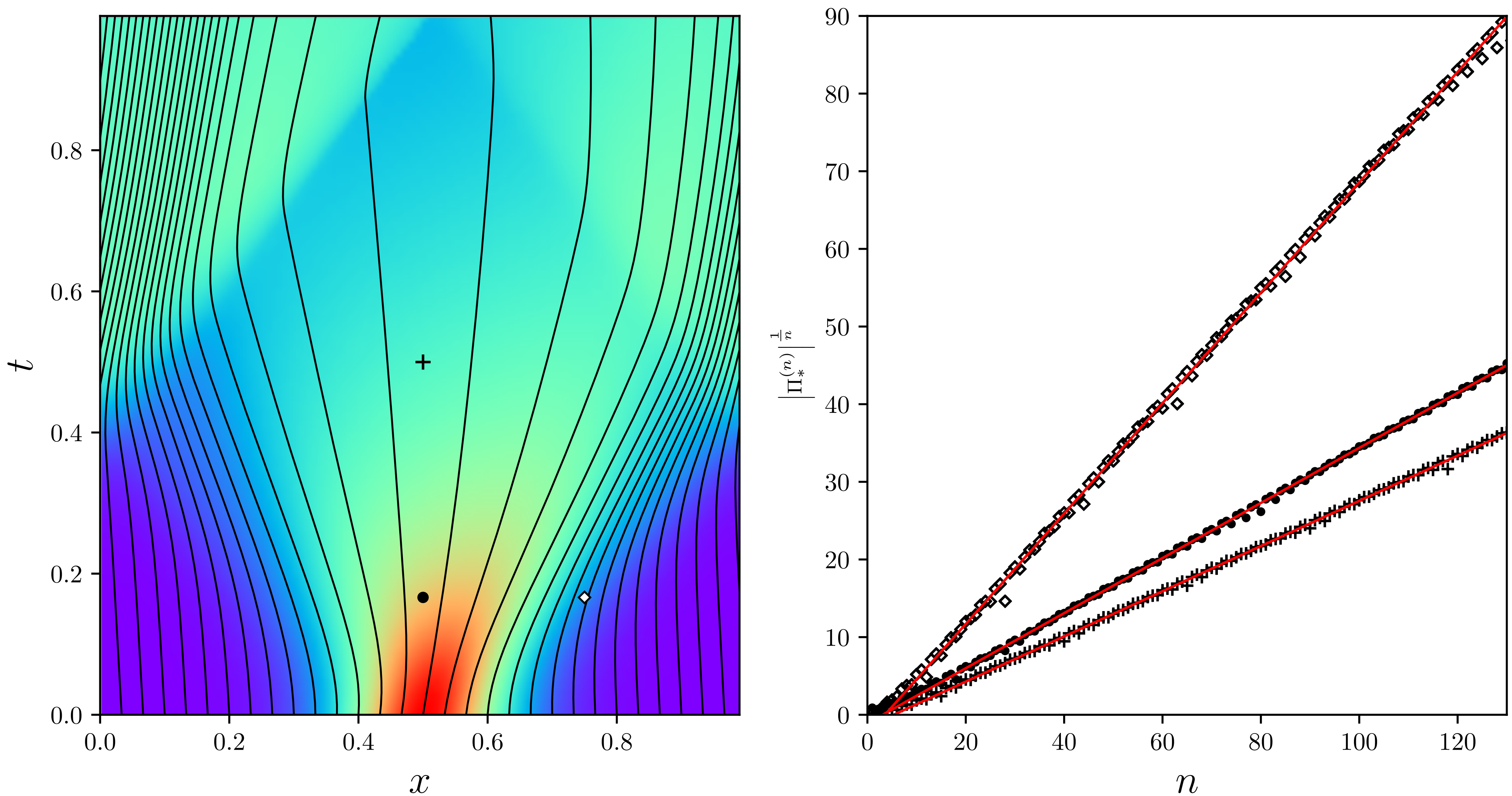
- Simply invert operator! $\left(1 + \tau_\Pi \mathcal{D}\right)$

$$\Pi^{\mu\nu}_{(n)} = \left(-\tau_\Pi \mathcal{D}\right)^{n-1} \left(-\eta \sigma^{\mu\nu}\right)$$

## **Nonlinear case** [Heller, Serantes, Spaliński, Svensson, BW (2021)]

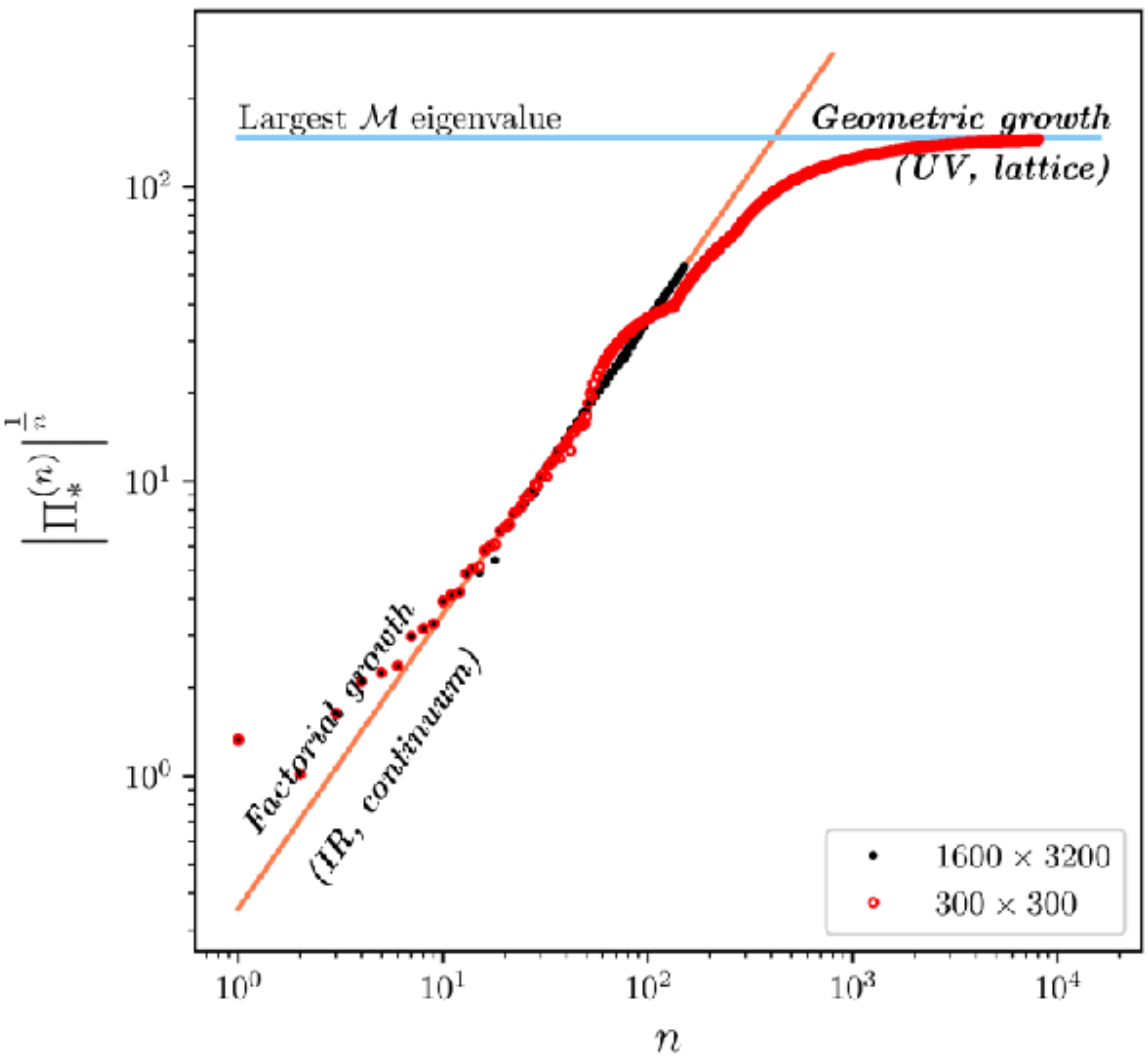
- Given a microscopic solution $T, U^{\mu}$ can easily evaluate

$$\Pi^{\mu\nu}_{(n)} = (-\tau_{\Pi}\mathcal{D})^{n-1}(-\eta\sigma^{\mu\nu})$$




- Factorial growth, divergent series, zero radius of convergence
- Generalises Bjorken flow-type results to 1+1 w/ no symmetry
- Confirmed across a variety of MIS-like models

- Lattice: $\left|\Pi^{\mu\nu}_{(n)}\right|^{\frac{1}{n}}$ saturates to largest eval of $\mathcal{D}$, set by inverse lattice spacing




- Generalises the linear result

## Analytic control through Singulants

[Heller, Serantes, Spaliński, Svensson, BW (to appear, 2021)]

- We have seen how expressions of the form

$$\Pi^{\mu\nu}_{(n)} = (-\tau_\Pi \mathcal{D})^{n-1} (-\eta\sigma^{\mu\nu})$$

lead to n! growth

- In fact, admit a large n analytic solution ~ n!

$$\Pi^{(n)}_{*}(t,x) \sim A(t,x)\frac{\Gamma(n+\beta(t,x))}{\chi(t,x)^{n+\beta(t,x)}},$$

[Dingle (1973)]

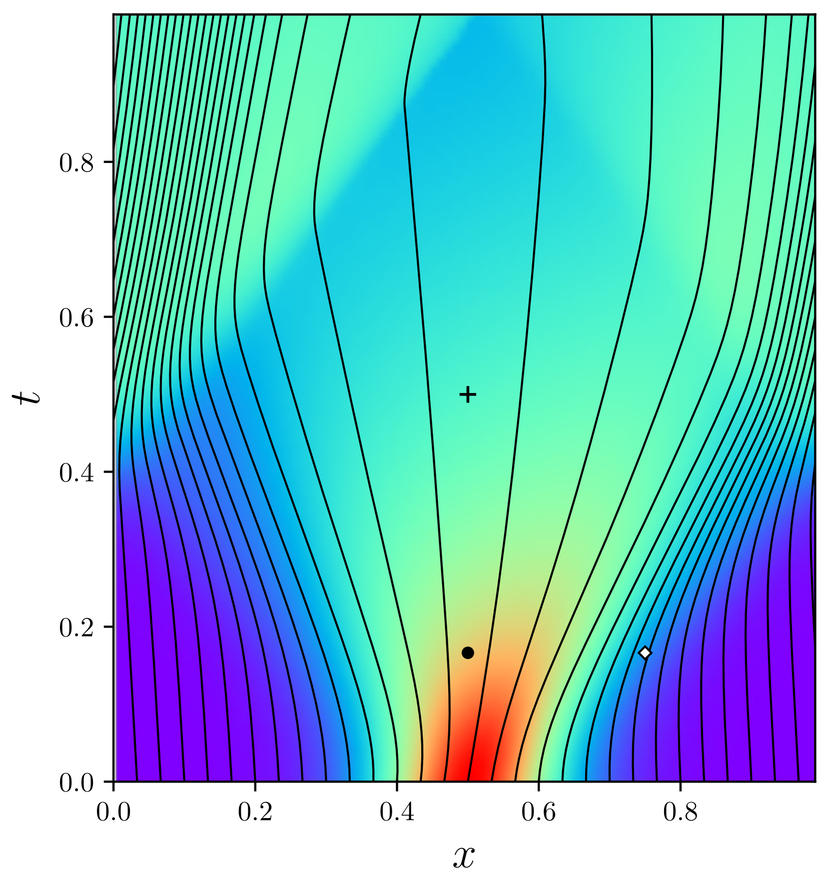
Provided the **singulant** field $\chi$ obeys a linear PDE

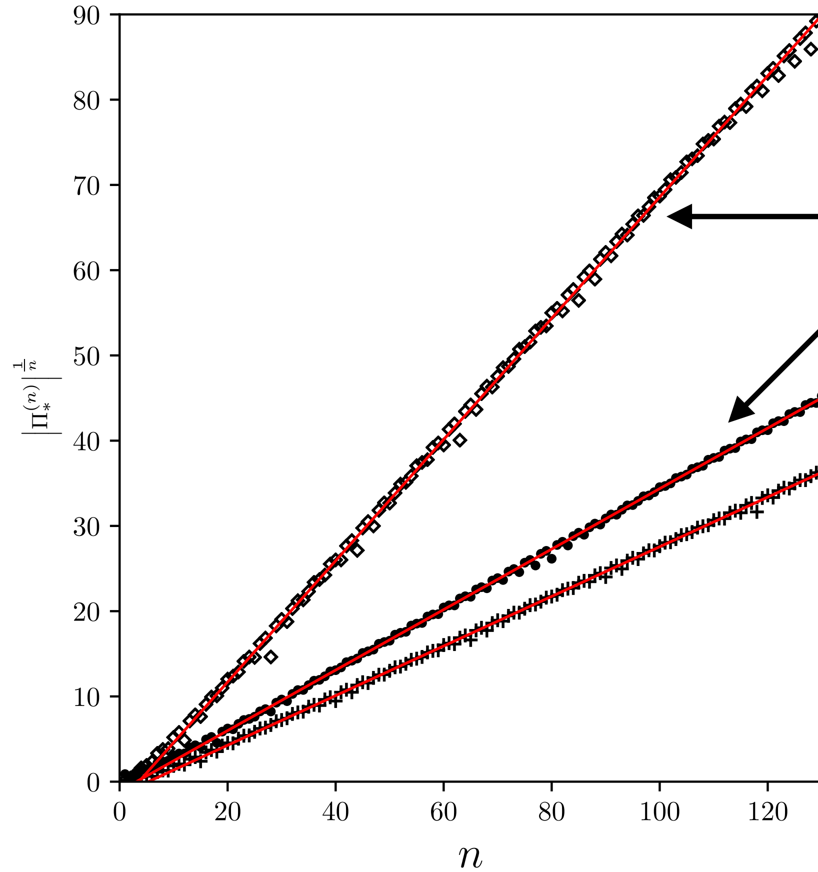
$$U^\mu \nabla_\mu \chi = \frac{1}{\tau_\Pi(T)}$$

- They can also be obtained with a WKB-type analysis

- We computed $\chi$ eoms in MIS/BRSSS, HJSW, DN, Holography, Kinetic Theory

# Analytic control through Singulants

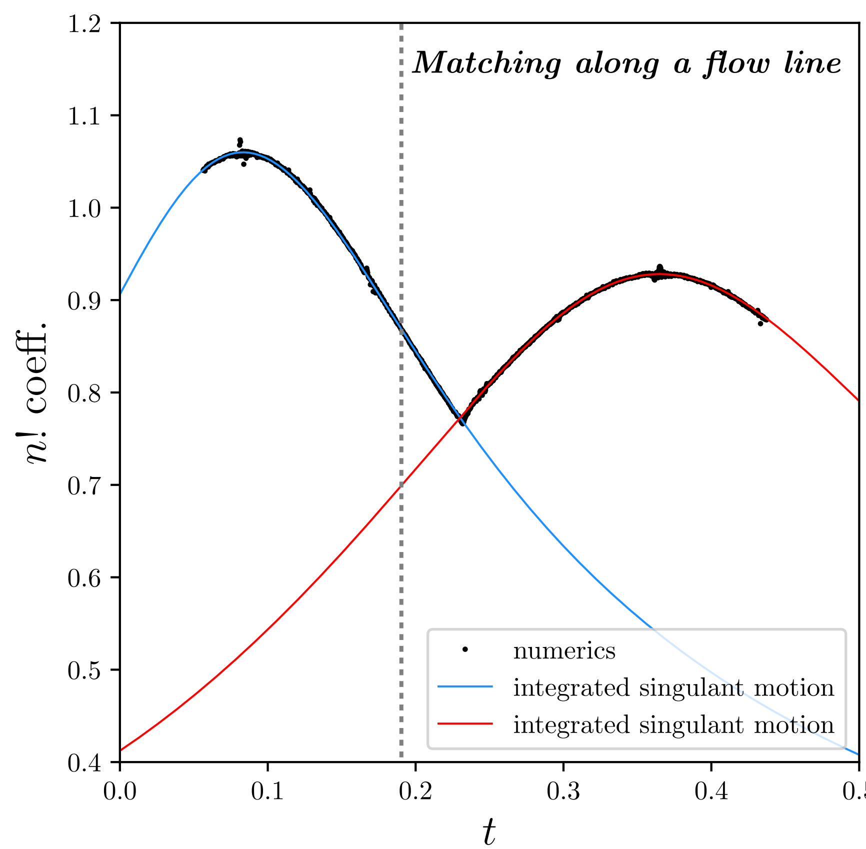


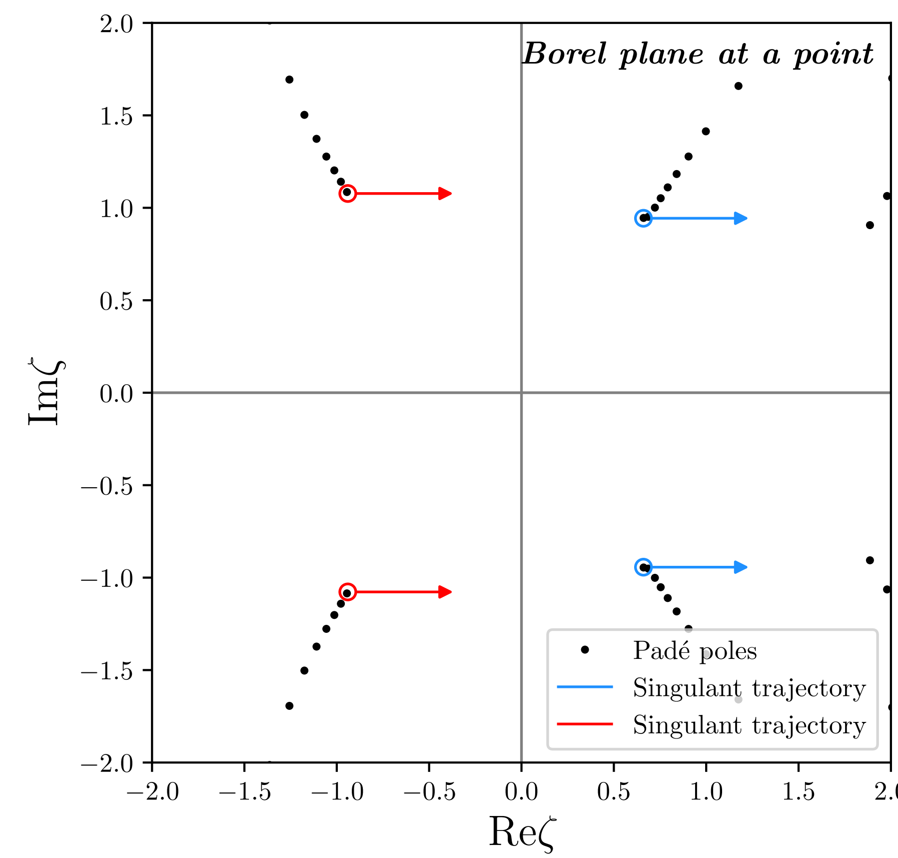




$$\left|\Pi_*^{(n)}\right|^{\frac{1}{n}} \sim \frac{n}{e|\chi|}$$

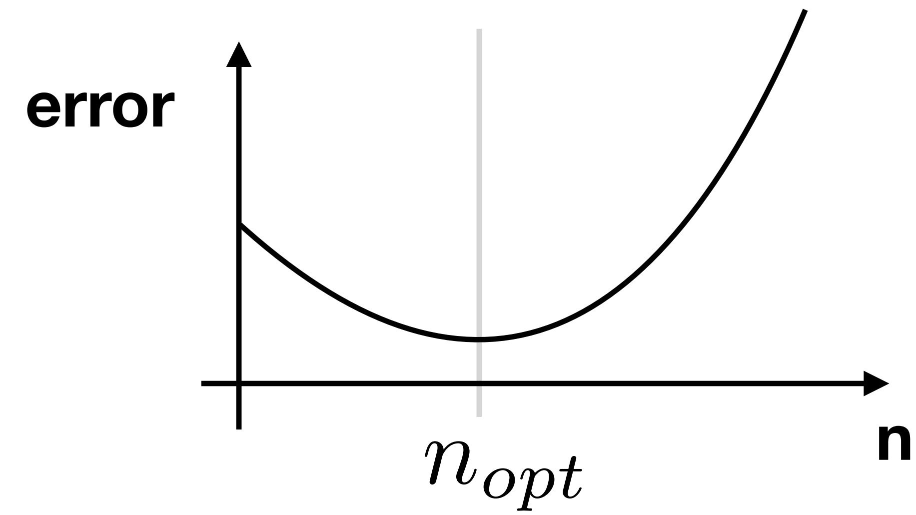
**Solve**

$$U^\mu \nabla_\mu \chi = \frac{1}{\tau_\Pi(T)}$$



## Analytic control through Singulants




### Some comments on $\chi$

- Control the order of optimal truncation

$$\Pi_*^{(n)}(t,x) \sim A(t,x)\frac{\Gamma(n+\beta(t,x))}{\chi(t,x)^{n+\beta(t,x)}},$$

$$\partial_n \left|\Pi_*^{(n)}\right|_{n=n_{opt}} = 0 \implies n_{opt} = |\chi| \quad \text{(large } |\chi|\text{)}$$

- Motion of $\chi$ controls 'hydrodynamisation' $U^\mu \nabla_\mu \chi = \dfrac{1}{\tau_\Pi(T)}$

- Can map singulant eom to a dispersion relation for a linear response problem

$$U^\mu \nabla_\mu \chi \to i\omega, \qquad e^\mu \nabla_\mu \chi \to \pm ik$$
$$U^\mu \nabla_\mu \chi = \frac{1}{\tau_\Pi(T)} \to \omega = -\frac{i}{\tau_\Pi(T)} = \omega_{NH}(k=0)$$

not QNMs!

# Summary

**Summary**

- Hydrodynamic phenomena are ubiquitous
- Fundamental links to black hole physics
- So far poorly understood as a gradient expansion

$$\Pi^{\mu\nu} = \sum_{n=1}^{\infty} \Pi^{\mu\nu}_{(n)} \left[T, U\right]$$

- **QNMs**
  - Finite radius set by singularity (collision with non-hydro qnm)

- **On-shell constitutive relations**
  - Generically divergent (numerics for 1+1 flows in MIS-like models)
  - Can be rendered geometric/convergent with momentum space cutoff

## Summary

**Singulants,** $\chi$

- Govern the large order hydrodynamic gradient expansion
- Obey simple linear equations, not unlike QNM equations
    - MIS/BRSSS, HJSW, DN, Holography, Kinetic Theory
- Coefficient of n! growth $\sim |\chi|^{-1}$
- Recede from origin over time ~ hydrodynamisation (order of optimal truncation $\sim |\chi|$ )
- They are not QNMs in general

## Some future directions

- **Divergence in holography**
  - We constructed singulant equations in holography,
  - But this is only a necessary condition for n! growth
  - Bjorken flow is one known example

- **Other EFTs**
  - Hydrodynamics can be viewed as a simple classical EFT, can we apply these techniques in other cases?

- **IVP well-posedness**
  - Orthogonal to our concerns so far
  - Seems generally incompatible with gradient expansion
  - Currently pursued solutions are to solve an unrelated problem e.g. IVP in 'toy' models: MIS, BRSSS, BDNK, …

**Thank you for your attention!**