Hydrodynamic Attractors in Holographic Bjorken Flow and Resurgence

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UA, HEP Seminar
10/22/21
Overview: Heavy Ion Collisions
Overview: Attractor

https://en.wikipedia.org/wiki/Limit_cycle
Overview: Attractor

\[ y'(x) = \cos(\pi x y(x)) \]
Overview: Attractor

[first: Heller, Spalinski 1503.07514]
Overview: Attractor
Overview: Unreasonable Effectiveness of Hydro at Early Times
Overview: Resurgence

- Divergent series
- Borel Transform
- Borel Resummation
- Poles in Borel plane
- Instantons
- Hydro expansion is divergent

\[ \epsilon = \frac{3}{8} N_c^2 \pi^2 \frac{1}{\tau^{4/3}} \left( \epsilon_2 + \epsilon_3 \frac{1}{\tau^{2/3}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \ldots \right) \]

[Heller, Spalinski 1302.0697]
Holographic Setup

5-dimensional anti-de Sitter space-time

4-dimensional space-time (hologram)

Black Hole

Superstrings

http://www.quantum-bits.org/?p=1134
Five-dimensional metric Ansatz

Einstein Gravity

$$ds^2 = 2dr dv - A(v, r)dv^2 + e^{B(v,r)} S(v, r)^2 (dx_1^2 + dx_2^2) + S(v, r)^2 e^{-2B(v,r)} d\xi^2,$$

$$\lim_{r \to \infty} \frac{1}{r^2} ds^2 = -d\tau^2 + dx_1^2 + dx_2^2 + \tau^2 d\xi^2,$$
Bjorken Flow

\[ \tau = \sqrt{t^2 - z^2} \]

\[ u^\mu = (u^t, u^x, u^y, u^z) = (t, 0, 0, z)/\sqrt{t^2 - z^2} \]

\[ \tau \partial_\tau \ln \epsilon = -\frac{4}{3} + \frac{16 C_\eta}{9 \tau T} + \frac{32 C_\eta C_\pi (1 - C'_\lambda)}{27 \tau^2 T^2} \]

\[ \frac{\Delta p}{\epsilon} \equiv \frac{P_T - P_L}{\epsilon} = 2 + \frac{3}{2} \frac{\partial_\tau \epsilon}{\epsilon} \]
Gravitational Setup

\[ ds^2 = 2dr dv - A(v, r) dv^2 + e^{B(v, r)} S(v, r)^2 (dx_1^2 + dx_2^2) + S(v, r)^2 e^{-2B(v, r)} d\xi^2, \]

Einstein Field Equations singular at horizon

\[ A(v, r) \longrightarrow \text{divergent} + A_s(v, r) \]

Same for S, B
Time Evolution Setup

\[ u = \frac{1}{r} \]

\[ ds^2 = 2drdv - A(v, r)dv^2 + e^{B(v, r)}S(v, r)^2(dx_1^2 + dx_2^2) + S(v, r)^2e^{-2B(v, r)}d\xi^2, \]

- Rewrite EFEs as: \[ \frac{d\Phi}{dt} = \mathcal{F}[\Phi], \quad \Phi = (t, a_4, B_s(u)) \]

- Start with initial conditions: \[ \Phi(t_0) = \{t_0, a_4(t_0), B_s(t_0, r)\} \]

- Time evolve \[ \Phi(t_0 + dt) = \Phi(t_0) + \Phi'(t_0)dt \]
# Nested DEQs Structure

- Rewrite EFEs as:

\[
\frac{d\Phi}{dt} = \mathcal{F}[\Phi] \quad \Phi = (t, a_4, B(u))
\]

<table>
<thead>
<tr>
<th>DEQ 1</th>
<th>0th derivatives</th>
<th>1st derivatives</th>
<th>2nd derivatives</th>
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<td>{As[t, u], Bs[t, u], Ss[t, u], dplusBs[t, u], dplusSs[t, u]}</td>
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<td>{As[^0,2][t, u]}</td>
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<tr>
<td>DEQ 5</td>
<td>{As[t, u], Ss[t, u], dplusBs[t, u], dplusSs[t, u]}</td>
<td>{As[^0,1][t, u]}</td>
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[Chesler, Yaffe 0812.2053]

https://github.com/BoGGoG/DEQSystemStructureVisualization
### Gravitational Setup

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- DEQ 1: Given $ Bs(u) $ solve for $ Ss(u) $  
- DEQ 2: Given $ Bs(u), Ss(u) $ solve for $ dplusSs(u) $  
- ...  
- Extra equations for $ \frac{da_4}{dt} \quad dB_s(t, u) $
Holography

\[ T_{\nu}^{\mu} = T_{\nu}^{\mu}(t, a_4, b_4) \]

- Time evolve 5D metric
- Extract metric components: \((a_4(t), b_4(t))\)
- Get field theory hydro quantities: \((\epsilon(t), P_T(t), P_L(t))\)
Holography: Energy Density Evolution
Holography: Pressures

$P_T$ vs $tT$

$P_L$ vs $tT$
Attractor: Energy Density Exponent $f$

Bjorken Hydro: \[ \epsilon \sim t^f(t) \]

\[ f(\tau) = \tau \partial_\tau \ln \epsilon = -\frac{4}{3} + \frac{16C_\eta}{9\tau \tilde{T}} + \frac{32C_\eta C_\pi (1 - C_\lambda)}{27\tau^2 \tilde{T}^2} \]
Attractor: Pressure Anisotropy \[ \Delta P = P_L - P_T \]
Attractor: Entropy Density

\[ \sigma(\tau) := \frac{\hat{S}(\tau)}{2\pi^4 T_{\text{ideal}}^3(\tau)} = \frac{A_{\text{AH}}(\tau)}{\pi^3 \Lambda^2 A} \]
Speed of Sound

\[ c^2 = \left( \frac{\partial P}{\partial \rho} \right)_s \]
Speed of Sound

\[ c^2 = \left( \frac{\partial P}{\partial \rho} \right)_s \]

How to get derivatives at constant entropy?

\[ d\varepsilon|_\sigma, \quad dP_{T/L}|_\sigma \]
Some Attempts of Calculating the Speed of Sound

\[ c^2 = \left( \frac{\partial P}{\partial \rho} \right)_s \]

- The naïve one:
  - completely ignores entropy
  - separately for every curve
  - late times: entropy = const \(\rightarrow\) correct

\[ \frac{dP}{d\epsilon} = \frac{dP}{dt} \left( \frac{d\epsilon}{dt} \right)^{-1} \]

- The “good intentions” idea:
  - also ignores entropy
  - introduces idea of derivative between curves
  - also good for late times
  - lead to the current procedure
Derivatives at $\sigma = \text{const}$ \quad $d\epsilon|_{\sigma}$, \quad $dP_{T/L}|_{\sigma}$

We have:

\[
\begin{array}{c}
\epsilon(t_i) \\
\epsilon(\sigma(t_i)) \\
P_{T/L}(t_i) \\
P_{T/L}(\sigma(t_i)) \\
\sigma(t_i)
\end{array}
\]

For close by initial conditions:

\[
\epsilon(t_0) = \{\epsilon_0 - n\delta\epsilon, \ldots, \epsilon_0, \ldots, \epsilon_0 + n\delta\epsilon\}
\]
Noronha Plot 1 (b): , $\Lambda$ from fitting $\epsilon$
Derivatives at $\sigma = \text{const}$
Derivatives at $\sigma = \text{const}$

- Interpolate every curve $\epsilon(\sigma)$ $\epsilon(\sigma)$
- Get $\epsilon(\sigma_i)$ for universal grid
- Take derivatives between curves for constant $\sigma_i$
- Do the inverse and get back to $t$
- Same for $P_{T/L}$
Speed of Sound

\[ c^2 = \left( \frac{\partial P}{\partial \rho} \right)_s \]

\[ d\varepsilon|_\sigma, \quad dP_T/L|_\sigma \]
Methods Comparison

Orange: Derivatives at constant $t$,
Green: const normalized entropy density
Red: $\frac{dp}{dt} \frac{dt}{d\epsilon}$
Future Ideas / Plans

- Speed of sound attractor
- Scan phase diagram
  - introduce chem. potential

Future Ideas / Plans

- Speed of sound attractor
- Scan phase diagram
  - introduce chem. Potential
- Magnetic Field
- Calculation of speed of sound from correlation functions
  5D metric -> fluctuation on top -> holography -> speed of sound
  (Quasinormal Modes)
- Resurgence
  - find attractors
  - speed of sound?
\begin{align}
A(v, z) &= z^2 A_s(v, z) + \xi(v)^2 + \frac{2\xi(v)}{z} + \frac{1}{z^2} \\
B(v, z) &= z^4 B_s(v, z) - \frac{2z^3}{9v^3} \left(3v^2 \xi(v)^2 + 3v \xi(v) + 1\right) + \frac{z^2(2v \xi(v) + 1)}{3v^2} - \frac{2z}{3v} - \frac{2 \log(v)}{3} \\
S(v, z) &= z^3 S_s(v, z) + \frac{3v \xi(v) + 1}{3v^{2/3}} + \frac{z^2(9v \xi(v) + 5)}{81v^{8/3}} - \frac{z}{9v^{5/3}} + \frac{v^{1/3}}{z}
\end{align}

\begin{align}
\bar{\kappa} \langle T_{00} \rangle &= \frac{1}{4} \left(2B^2 \log(\Lambda L) - 3a_4(\tau)\right) \\
\bar{\kappa} \langle T_{11} \rangle &= \langle T_{22} \rangle = -\frac{a_4(\tau)}{4} + b_4(\tau) + \frac{1}{12} \left(-\frac{2}{\tau^4} + 6B^2 \log(\Lambda L) - 3B^2\right) \\
\bar{\kappa} \langle T_{33} \rangle &= -\frac{3\tau^4}{12\tau^2} \left(a_4(\tau) + 8b_4(\tau)\right) - 6\tau^4 B^2 \log(\Lambda L) + 4
\end{align}