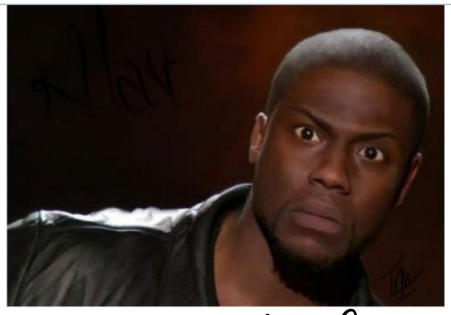
Reminder: My a's are written as "a", my d has a tail d

tail!

Lets begin by ignoring why we would want to do any of this in the first place!

Your Face as you read That first sentence →



Don't Fret. we will come back to this!

Instead I will tell you our Goal.

Primary resource: arXiv:1712.07903

Goal:
- Introduce Random matricies
- Compute pdf of eigenvalues
- Associate pdf with Partition fct of 2D Coulomb gas
- Describe where this has resurfaced in Physics.

Warm up! Consider Populating, H, an $N \times N$ matrix with random values drawn from a gaussian probability density function with mean $\mu = 0$ and variance $\sigma^2 = 1$

Such a matrix will not have any symmetries.

But we can make it so!

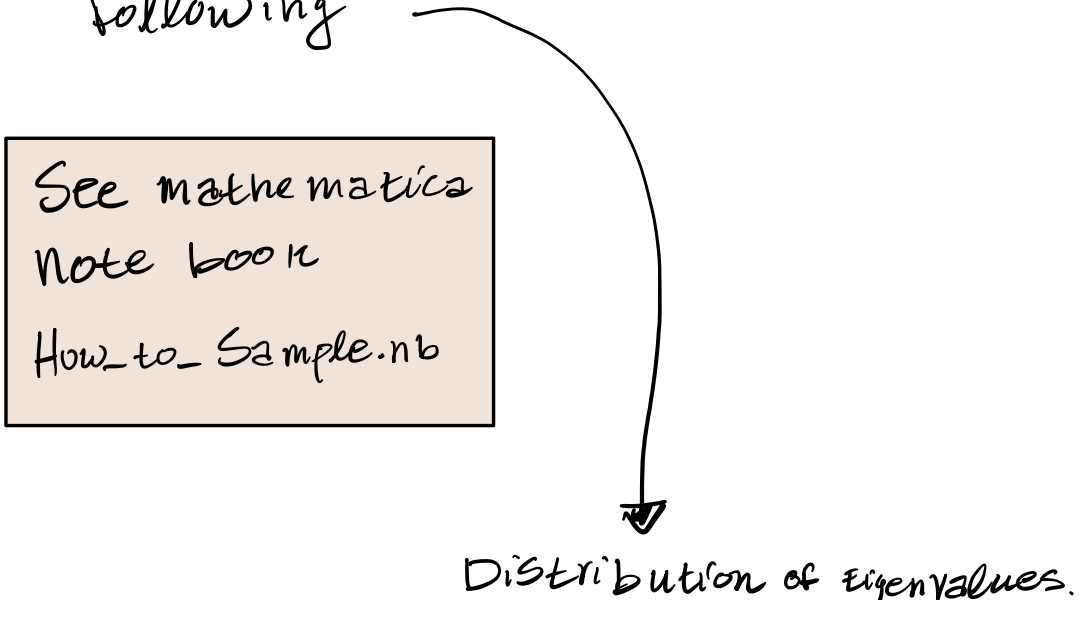
$\tilde{H} = \frac{1}{2}(H + H^T)$ - Gaussian orthogonal ensemble!

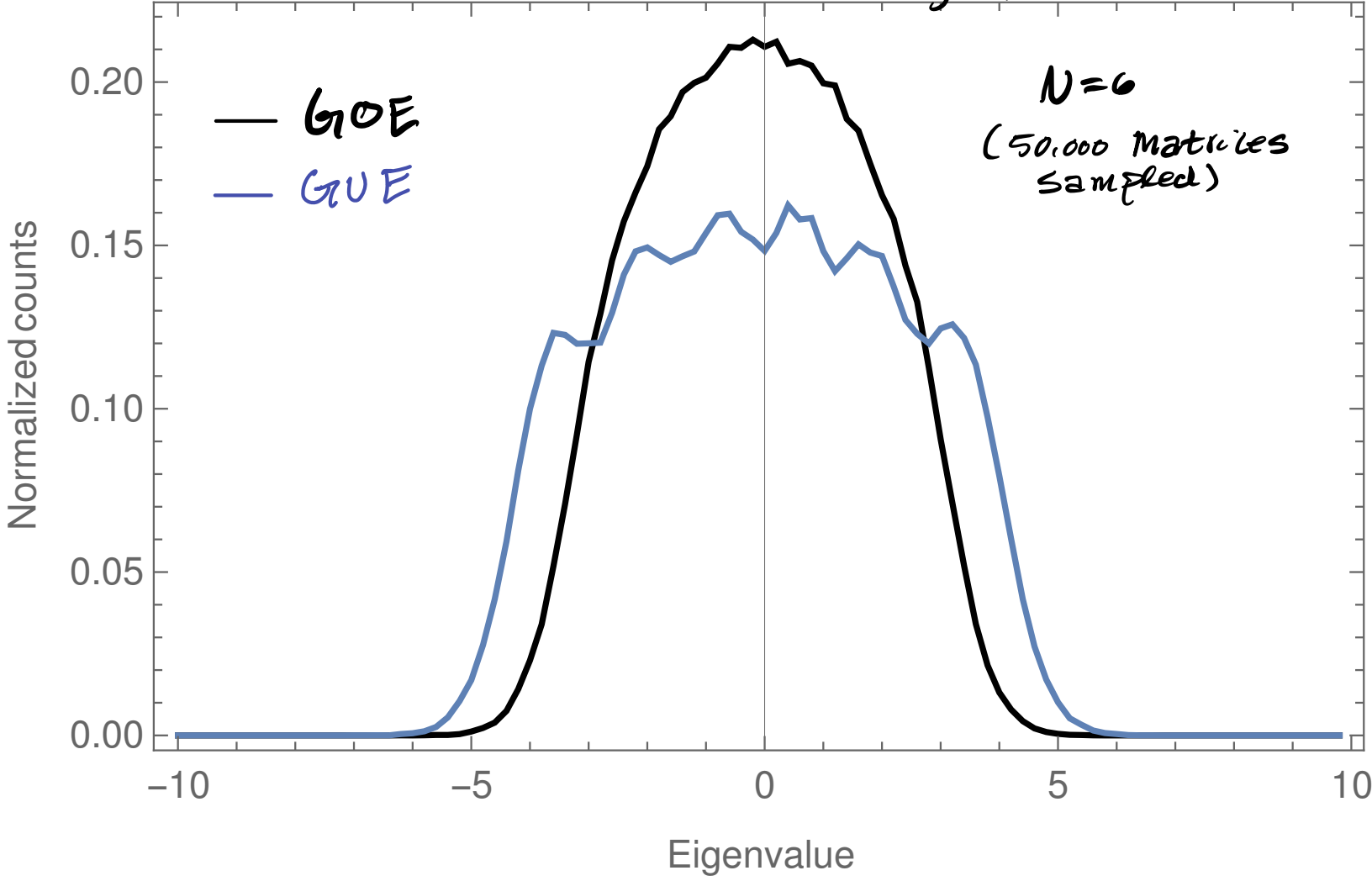
or we could choose to have complex entries in which case it would be best to

$\tilde{H} = \frac{1}{2}(H + H^\dagger)$, $\tilde{H}^\dagger = \frac{1}{2}(H^\dagger + H)$ ✓

- Gaussian unitary Ensemble.

- IF we were to compute the eigenvalues of a large number (say 50000) of these matrices we would find the following




GOE
GUE
N=6
(50,000 matrices sampled)
Normalized counts
0.20
0.15
0.10
0.05
0.00
−10
−5
0
5
10
Eigenvalue

While it may be interesting to study a single instance it will actually be much more fruitful to study the statistic behavior of an ensemble of such matrices.

- You might ask. Can we predict the shape of the above curves? The answer is yes! (Though I wont do it here)
- And why is this useful to a physicist?
  ↳ I'll answer this as we go on.

Some info about Statistics

- pdf - $\rho(x)dx$ likelyhood your varible is with $(x, x+dx)$
- normalized such that $\int dx\, \rho(x) = 1$
- cdf - Cumulative distribution Function

$$F(x) = \int_{-\infty}^{x} \rho(x)dx$$

The probability that x (your random variable) is less Then or equal to x.

- mean $\langle x \rangle = \int dx\, x\, \rho(x)$
- moments $\langle x^n \rangle = \int dx\, x^n \rho(x)$
- Variance $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$ (uncertainty)

In QM Parlance

- Joint pdf $\rho(\vec{x}) = \rho(x_1\, x_2 \cdots x_n)$

- Probability that $a_i < x_i < b_i$

$$P = \left(\prod_{i=1}^{n} \int_{a_i}^{b_i} dx_i\right) \rho(\vec{x})$$

- Factorization $\rho(\vec{x}) = \prod_{i=1}^{n} \rho_i(x_i)$

For x_i independent

- marginal pdf

$$\rho(x_1) = \prod_{i=2}^{n} \int_{a_i}^{b_i} dx_i\, \rho(x_2 \cdots x_n)$$

How can we write down the pdf of the N^2 entries of a random matrix H?

each entry of H is an inde. Gaussian variable. Therefore ρ factorizes

$$\rho = \prod_{i,j=1}^{N} \rho_{ij}(H_{ij}), \quad \text{each } \rho_{ij} = \frac{1}{\sqrt{2\pi}} e^{-\frac{H_{ij}^2}{2}}$$

$$\Rightarrow \rho = \prod_{\substack{i=1 \\ j=1}}^{N} \frac{1}{\sqrt{2\pi}} e^{-H_{ij}^2/2}$$

How can we write down the pdf of the $\frac{N(N+1)}{2}$ entries of a random matrix H_s?

From the previous question

$$\rho = \prod_{\substack{i=1 \\ j=1}}^{N} \frac{1}{\sqrt{2\pi}} e^{-H_{ij}^2/2}$$

split this up.

$$= \left(\prod_{i=j=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-H_{ii}^2/2} \right) \left(\prod_{i<j} \frac{1}{\sqrt{2\pi}} e^{-H_{ij}^2/2} \right) \left(\prod_{i>j} \frac{1}{\sqrt{2\pi}} e^{-H_{ij}^2/2} \right)$$

diagonal — upper triangle — lower triangle

The upper and lower triangle will have the same entries for our symmetric matrices hence

$$\rho = \left(\prod_{i=j=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-H_{ii}^2/2} \right) \left(\prod_{i<j} \frac{1}{\sqrt{\pi}} e^{-H_{ij}^2} \right)$$

Notice the different factors of two!

Now an important question

Can we rewrite this quantity in terms of the eigenvalues rather then entries?

That is we want something like

$$\rho(H_n \cdots)\, dH = \rho(H(x,O), \cdots)\, J(H \to (O,x))\, dO\, \Pi\, dx$$

H is real sym. Hence it can be diagonalized

$$H = O X O^T, \quad O \text{ - orthogonal ie. } O O^T = 1$$
$$X = \mathrm{diag}(x_1, x_2 \cdots)$$

$$\Rightarrow \quad \delta H = \delta O X O^T + O \delta X O^T + O X \delta(O^T)$$

Note: $\delta(O O^T) = \delta(1)$

$$\delta O O^T + O \delta O^T = 0 \quad \therefore \quad \delta O^T = -O^T \delta O O^T$$

$$\delta H = \delta O X O^T + O \delta X O^T + O X (-O^T \delta O O^T)$$
$$= O O^T \delta O X O^T + O \delta X O^T - O X O^T \delta O O^T$$
$$= O [O^T \delta O X + \delta X - X O^T \delta O] O^T$$

$\delta H = O\, \delta \hat{H}\, O^T$, with $\delta \hat{H} = O^T \delta O X + \delta X - X O^T \delta O$

define: $\delta \Omega = O^T \delta O$

$$\Rightarrow \delta \hat{H} = \delta\Omega X - X \delta\Omega + \delta X$$
$$= [\delta\Omega, X] + \delta X$$

Consider:

$$(O^T \delta O)^T = (\delta O)^T O$$
$$= \delta(O^T) O = -O^T \delta O O^T O = -O^T \delta O$$

$\therefore \delta\Omega$ is antisymmetric

Counting: H has $\frac{N(N+1)}{2}$ components

O has $\frac{N(N-1)}{2}$ components

x has N components.

$[H] = [O] + [x]$ ✓

In terms of components.

$$d\hat{H}_{ij} = d\Omega_{ij}(x_j - x_i) + dx_i \delta_{ij}$$

hence $\frac{d\hat{H}_{ij}}{d\Omega_{kl}} = \delta_{ik}\delta_{jl}(x_j - x_i)$ $\frac{d\hat{H}_{ij}}{dx_k} = \delta_{ik}\delta_{ij}$

How do we get the determinant?

Explicitly in $N=3$

Consider the Form $(d\Omega_{ij}, x_j)$

$(d\Omega_{12}, d\Omega_{23}, d\Omega_{13}, x_1, x_2, x_3)$

H_{11}
H_{12}
H_{13}
H_{22}
H_{23}
H_{33}

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ x_2-x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3-x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & x_3-x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\therefore$

$$\det J = -1\,(x_2-x_1)(x_3-x_1)(+1)(-x_3+x_2)(-1)$$
$$= (x_1-x_2)(x_1-x_3)(x_2-x_3)$$
$$= -(x_1-x_2)(x_3-x_1)(x_2-x_3)$$

Hence For N in General

$$\det J = \prod_{j<n}^{N} (x_j - x_n)$$

Note on orderings:

There are 2^N distinct combinations of N elements. And $N!$ arrangements or orderings of such elements.

Now returning to our Problem

$$\rho(H)dH = \rho(H(x,\theta))\,|J|\,d\theta\,dx$$

$$\rho(H) = \prod_{i=1}^{N} \frac{1}{(2\pi)^{1/2}} e^{-H_{ii}/2} \prod_{i<j}^{N} \frac{1}{\pi^{1/2}} e^{-H_{ij}}$$

Quick note:

$$\mathrm{Tr}\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}^2 = \mathrm{Tr}\begin{pmatrix} H_{11}^2 + H_{12}^2 & H_{11}H_{12} + H_{12}H_{22} \\ H_{11}H_{12} + H_{22}H_{12} & H_{12}^2 + H_{22}^2 \end{pmatrix}$$

$$= H_{11}^2 + H_{22}^2 + 2H_{12}^2$$

Therefore we can rewrite as...

$$\rho = \frac{1}{(2\pi)^{N/2}} \frac{1}{\pi^{\frac{N(N-1)}{4}}} e^{-\frac{1}{2}\mathrm{Tr}H^2}$$

Note also: $\mathrm{Tr}(H^2) = \mathrm{Tr}(HH) = \mathrm{Tr}(O x O^T O x O^T)$

$$= \mathrm{T}(O x^2 O^T) = \mathrm{Tr}(x^2)$$

$$= \frac{1}{(2\pi)^{N/2}} \frac{1}{\pi^{\frac{N(N-1)}{4}}} e^{-\frac{1}{2}\mathrm{Tr}x^2} \prod_{i<j}^{N} |x_i - x_j| \, dO \, dx$$

Now we just are left with integrating out dO i.e. we want the volume of the space of orthogonal matrices.

- I will simply quote the result since this is a seminar's worth of material in its own right.

- The set of matrices O which satisfy $OO^T = I$ form a $\frac{N(N-1)}{2}$ dim. submanifold of $\mathbb{R}^{N^2}$ called the stiefel manifold V_N

$$\text{Vol}(V_O) = \int_{V_O} dO = \frac{2^N \pi^{N^2/2}}{\Gamma_N(N/2)}$$

(See. 1509.00537)

For $\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma(a - (i-1)/2)$

Finally we can integrate out O to obtain the marginal pdf of the eigenvalues

$$\int_{V_O} \frac{1}{(2\pi)^{N/2}} \frac{1}{\pi^{\frac{N(N-1)}{4}}} e^{-\frac{1}{2}\text{Tr}x^2} \prod_{i<j}^{N} |x_i - x_j| \, dO \, dx$$

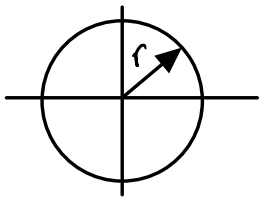
$$= \frac{2^N \pi^{N^2/2}}{\Gamma_N(N/2)} \frac{1}{(2\pi)^{N/2}} \frac{1}{\pi^{\frac{N(N-1)}{4}}} e^{-\frac{1}{2}\text{Tr}x^2} \prod_{i<j}^{N} |x_i - x_j| \, dx$$

observations:

1. $e^{-\frac{1}{2}\pi x^2}$ this factor kills large eigenvalues. i.e. eigenvalues tend to be clustered near the origin

2. $\prod_{i<j}^{N} |x_i - x_j|$ on the other hand this factor stops eigenvalues from being the same. i.e. it is repulsive

3. Recall EM in 2D. Gaussian surface is S^1 (instead of S^2)



$\nabla \cdot E = \rho/\epsilon$ (Gauss' law)

$\int \vec{E} \cdot d\vec{A} = q/\epsilon$

$E \cdot 2\pi r = q/\epsilon$

$E = q/2\pi\epsilon r$

Potential: $-\nabla \cdot \phi(r) = E$

$-2\pi\phi = q/2\pi\epsilon r$

Logarithmic Potential. $\longrightarrow$ $\phi = -\frac{q}{2\pi\epsilon} \ln r$

$$\prod_{i<j}^{N} |x_i - x_j| = e^{\sum_{i<j} \ln(x_i - x_j)}$$

Some brief Algebra:

$$\frac{2^N \pi^{N^2/2}}{\Gamma_N(N/2)} \frac{1}{2^{N/2}\pi^{N/2}} \frac{1}{\pi^{N^2/4}\pi^{-N/4}} = \pi^{\frac{N(N-1)}{4}} \frac{2^{N/2}}{\Gamma_N(N/2)}$$

$$\Gamma_M(a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma(a-(i-1)/2)$$

$$= 2^{N/2} \pi^{\frac{N(N-1)}{4}} \cdot \frac{1}{\pi^{\frac{N(N-1)}{4}} \prod_{i=1}^{N} \Gamma\left(\frac{N}{2} - \frac{(i-1)}{2}\right)}$$

$$= \frac{2^{N/2}}{\prod_{i=1}^{N} \Gamma\left(\frac{N}{2} - \frac{(i-1)}{2}\right)}$$

$$= \frac{2^{N/2}}{\prod_{i=1}^{N} \Gamma\left(\frac{i}{2}\right)} = \frac{1}{Z}$$

$$\prod_{i=1}^{N} \Gamma\left(\frac{N}{2} - \frac{(i-1)}{2}\right) = \Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-1}{2}\right)\cdots\Gamma\left(\frac{1}{2}\right)$$
$$= \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{2}\right)\cdots\Gamma\left(\frac{N}{2}\right)$$
$$= \prod_{i=1}^{N} \Gamma\left(\frac{i}{2}\right)$$

we arrive at our final expression

$$\rho = \frac{1}{Z} e^{-\frac{1}{2}\Gamma r x^2 + \sum_{i<j} \ln|x_i - x_j|}$$

- This is nothing but the Partition function of a static gas of Charged Particles in an

attractive quadratic potential

$$- \sum_{i<j} \ln|x_i - x_j| = \frac{1}{2} \sum_{i \neq j} \ln|x_i - x_j|$$

$$= W - \text{The self energy}$$

- The attrative quadratic potential must balance against the energy required to create this configuration!

Note: scale $x \to \sqrt{N}\, x$

$$\rho \to \rho = \frac{1}{Z'} e^{-N^2 V(x)}, \quad V(x) = +\frac{1}{2}\frac{1}{N}\sum_i x_i^2 - \frac{1}{N^2}\sum_{i<j} \log(|x_i - x_j|)$$

- $N \to \infty$ limit is a Thermodynamic limit.

- one can compute Thermodynamic quantities such as the entropy which gives the Shannon entropy in this case.

Example:

Consider $N=2$, what is the distribution of the spacing between two eigenvalues?

Note: we must be careful in the analysis above we have over counted by $2^N N!$ So we must begin with

$$\rho = \frac{1}{2^N N! Z} e^{-\mathrm{Tr} X^2 + \frac{1}{2}\sum_{i \neq j} \ln|x_i - x_j|}$$

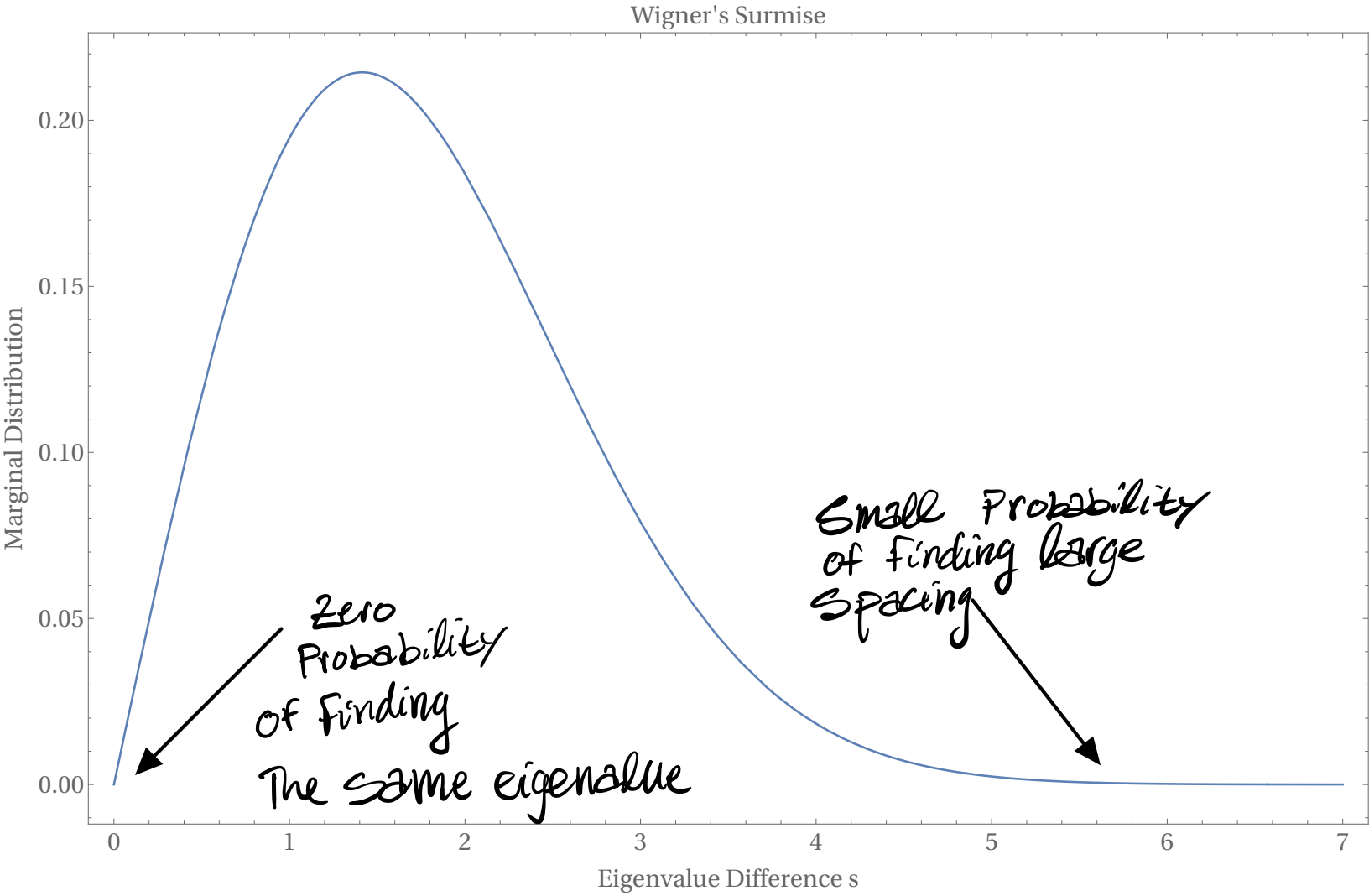
- Solution: introduce $s = x_1 - x_2$

we want the maginal distribution

$$\rho(s) = \int_{-\infty}^{\infty} dx_2 \, \frac{1}{2^2 2!} \frac{1}{Z(2)} e^{-\frac{1}{2}(x_2^2 + (s+x_2)^2) + \log s}$$

$$= \frac{1}{4} e^{-s^2/4} \cdot s$$

This result is known as Wigners Surmise.


Wigner's Surmise
Marginal Distribution
0.20
0.15
0.10
0.05
0.00
0
1
2
3
4
5
6
7
Eigenvalue Difference s
Zero Probability of Finding The same eigenvalue
Small Probability of finding large Spacing

History: 1950's

- Studying the energy spectra of heavy nuclei.
- Very complex. Led to statistical approach
- Urban legend: At a conference on Neutron Physics by time of flight at Oak Ridge in 1956 A topic of discussion was the possible shape of the distribution of the spacings of the energy levels of a heavy nucleus. Eugene Wigner apparently walked up to the chalk board and surmised that

$$\rho = \frac{1}{4} s e^{-s^2/4}$$

Modern uses:

arXiv: 1903.11115 "JT Gravity as a matrix Integral"

- Very interesting paper
- $NAdS_2/NCFT_1$: Partition function corresponds to genus expansion of Matrix integrals.
- Calculate: correlations of JT (Euclidean) Path int. $\langle Z(\beta_1) \cdots Z(\beta_n) \rangle$

corresponding to surfaces with n schwarzian boundaries. Computable via double scaling limit of certain matrix integrals.

topological expansion (matrix int.) = handles JT Gravity

- In the view of Perhaps the AdS is not dual to CFT but to an ensemble average of CFT's.

another example: Chaos Theory

See for example arXiv: 1706.05400

- late time sys. model is described by RMT
- Random matrix eigenvalue stats. considered as hallmark of quantum chaos.
  quantized classically chaotic system → spectral stats. of random matrix ensemble
- This work in particular: explicit analytic formula relating averages of OTOC and spectral form factor $R = \langle |Z(\beta, t)|^2 \rangle$
  $Z = \mathrm{Tr}\, e^{-\beta H - iHt}$