QUALIFYING EXAMINATION

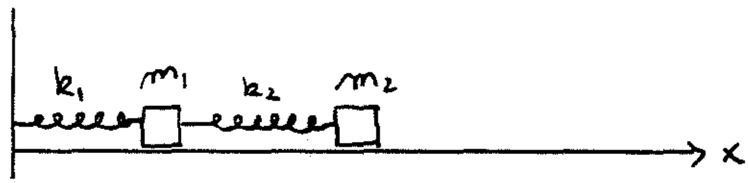
SPRING 1994

General Instructions: No reference materials (other than a calculator) are permitted. Do all work in your answer booklet. Turn in the questions for each part with the answer booklet. You may finish Part I early, turn it in and start work on Part II.

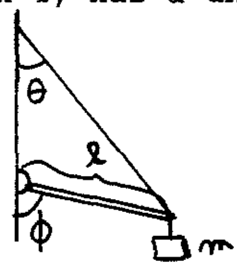
PART I: Classical Mechanics

Work any 5 of the 6 problems.

1. Write down the Lagrangian and find the equations of motion for the system below where the motion is constrained to the x-axis. Assume that the equilibrium lengths and force constants of the first and second springs are ℓ_1, k_1 and ℓ_2, k_2 respectively. Neglect friction.



2. Find the tension, T, in the light string in the diagram below if the rod, of mass M and length ℓ, has a uniform mass distribution. The mass being supported is m.



3. A bullet of mass 1.0 g is shot into a wooden pendulum bob of mass 1.0 kg. The bullet sticks in the bob after the collision and the bob rises vertically 5.0 cm after the collision. How much energy is lost as a result of the collision?

4. A bug of mass m crawls with constant speed along the spoke of a wheel which rotates with constant angular velocity ω about a vertical axis. Find the net force acting on the bug. Assume the spoke makes an angle ωt with the x-axis and the bug starts at the origin at t = 0. How is the force related to the centrifugal and Coriolis "forces" which would be seen by an observer rotating with the wheel?

5. A particle moving in a central field travels in a spiral orbit given by $r = a*\exp(k\theta)$. r and θ are plane polar coordinates and a and k are constants. Find the force and how θ varies with t.

6. Find the condition for stable circular orbits in a central field whose force is of the form ($k > 0$, $\epsilon > 0$)

$$f(r) = -k/r^2 - \epsilon/r^4.$$

PART II -- ELECTRICITY AND MAGNETISM

Work any 5 of the 6 problems. You may use either SI or Gaussian (cgs) units.

SI Constants

$\epsilon_o = 8.85\text{x}10^{-12}\ C^2/Nm^2$ $\mu_o = 4\pi\text{x}10^{-7}\ N/A^2$ $m_e = 9.1\text{x}10^{-31}$ kg

1. An infinitely long cylinder of radius b centered on the z axis is uniformly charged with a charge per unit length λ. What is the electric field at arbitrary distances ρ from the z axis? Note that the volume charge density is constant inside the cylinder.

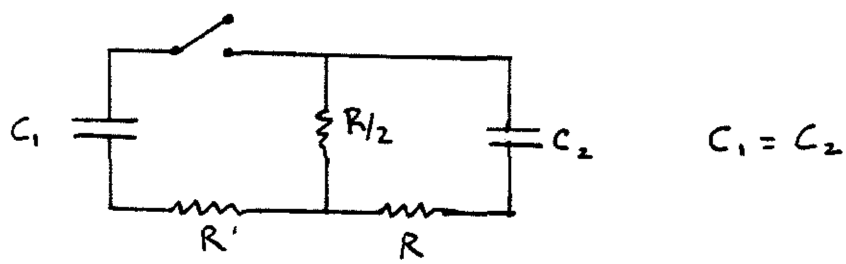
2. What work is done in moving a charge of 1 Coulomb from the point (R,0,0) to the point $(R/\sqrt{2},R/\sqrt{2},0)$ along a circular path in the xy plane in the presence of an electric field

$$\vec{E}(x,y,z) = V_o(x\ \hat{e}_y + y\ \hat{e}_x)/R^2$$

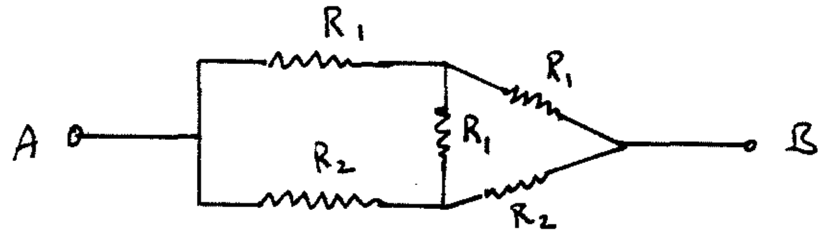
if V_o is 1 Volt and R is 1 meter?

3. A static electric field $\vec{E}$ in vacuum makes an angle θ with the normal to the planar surface of a material with dielectric constant ϵ. Inside the material the electric field has magnitude E′ and makes an angle θ' with the normal. What are E′ and θ' in terms of E, θ, and ϵ?

4. A switch is closed at t = 0 to discharge a capacitor $C_1(2\mu F)$ with initial charge q_0 in the circuit below when R = 5Ω and $C_2 = C_1$. At what time is the charge on C_1 equal to $q_0/10$?



5. Three resistors of resistance R_1 and two of resistance R_2 are connected as shown. What is the resistance R_T between points A and B?



6. A wire of resistance R makes a circular loop of radius a in the xy plane. A uniform magnetic induction of initial magnitude B(0) points in the z direction. The B field then decays to zero according to the relation

$$B(t) = B(0)e^{-\gamma t} .$$

What amount of heat energy is generated in the wire?

PART III -- QUANTUM MECHANICS

Answer any 5 questions.

1. A large number of systems are prepared identically, each system consisting of a particle moving in one dimension ($-\infty < x < \infty$) under the influence of a restoring force. The wave function $\Psi(x,t)$ describing each system may be written as a superposition of two stationary states

$$\Psi(x,t) = 0.5i\ \psi_1(x)e^{-i\alpha t} + (0.707 - 0.5i)\ \psi_2(x)e^{-i\beta t}$$

where ψ_1 and ψ_2 are orthonormal functions and α and β are constants. Assume that both ψ_1 and ψ_2 are even functions of x, i.e. $\psi(x) = \psi(-x)$. Express your answers to the following questions in terms of α, β, and physical constants, and where necessary, in terms of integrals involving ψ_1 and ψ_2.

(a) If the energy of one of the particles is measured, what are the possible outcomes and the probability of each?

(b) Suppose that the position of each particle is measured at time t. What is the standard deviation of the results? Are interference effects evident in the standard deviation? If so, how?

2. Three identical non-interacting spin-$\frac{1}{2}$ particles with mass m are confined to a one-dimensional box of width a.

(a) What is the ground state energy of the three particle system?

(b) Suppose that the particles are in a state such that the total spin quantum number is 3/2. What is the lowest energy allowed to the system in this case?

(c) For the case where the particles are in a state such that the total spin quantum number is 3/2 and the energy is the minimum allowed, write down explicitly the space part of the wave function in terms of the relevant single-particle wavefunctions.

3. A neurtron with magnetic moment $\vec{\mu}$ is immersed in a uniform magnetic field such that

$$\vec{B} = B_o(\hat{x} \sin\theta + \hat{z} \cos\theta)$$

(a) Find the stationary states in terms of eigenstates of S_z and corresponding energies for this system.

(b) For each of the stationary states found, evaluate the probability that a measurement of the neutron's spin component along the z-axis will yeild $\hbar/2$.

For the spin operators take

$$S_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The following identities may be be useful for simplifying expressions:

$$1 - \cos\theta = 2\sin^2 \frac{\theta}{2}$$

$$1 + \cos\theta = 2\cos^2 \frac{\theta}{2}$$

4. Two particles of mass m_1 and m_2, respectively, moving in one dimension are bound to each other through a force described by the potential energy function

$$V(x_1,x_2) = 0, \; a < |x_1 - x_2| < b$$

$$V(x_1,x_2) = \infty, \text{ elsewhere}$$

(a) Choose a convenient set of coordinates, write down the time-independent Schrodinger's equation governing this system in terms of those coordinates, and then solve it to obtain the allowed energies for this system relative to the energy due to the motion of the center of mass.

(b) Each energy level of this system is actually two-fold degenerate. Explain why.

The following identities may be useful

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$$

5. A particle of mass m is attached to one end of a rigid rod of negligible mass and length R. The other end of the rod rotates in the x-y plane about a bearing located at the origin.

(a) Taking the potential energy to be zero, write down the total energy of the system in terms of its angular momentum and show that, by introducing the appropriate operators in place of the classical quantities, Schrodinger's equation for this system can be written as

$$-\frac{\hbar^2}{2I}\frac{\partial^2\Psi}{\partial\theta^2} = i\hbar\frac{\partial\Psi}{\partial t}$$

where $I=mR^2$ is the moment of inertia. (Hint: the operator corresponding

to L_z is $\frac{\hbar}{i}\ \partial/\partial\phi$.)

(b) Solve Schrodinger's equation for this case to obtain the stationary states of this system and corresponding energies.

6. Answer the following questions in the context of one-electron atoms.

(a) Neglecting all forces but the Coulomb attraction, what is the degeneracy of each allowed energy level?

(b) Explain the origin of the spin-orbit interaction.

(c) In the presence of the spin-orbit interaction, describe a set of suitable quantum numbers for completely specifying a stationary state of the atom. Include in your description the allowed values for each quantum number atom.

(d) The energies of states with orbital quantum number $\ell = 0$ do not depend on the spin-orbit interaction. Is it also the case that the energies of these states are unaffected by the introduction of an external magnetic field? Why or why not?

PART IV -- MIXED TOPICS

Do problems from 4 of the 5 sections. Astrophysics may be chosen only in place of electronics. Do 6 of the 8 problems with at least one from each of the 4 chosen sections. Use a different answer book for each lettered section.

A: Relativity

1. A rocket (A) is launched at a speed $\frac{c}{2}$ along the y_1 axis in a stationary frame S_1. Meanwhile, another rocket (B) at rest in a frame S_2, is moving left at a speed $\frac{c}{4}$:

Using the Loreatz transformation equations:

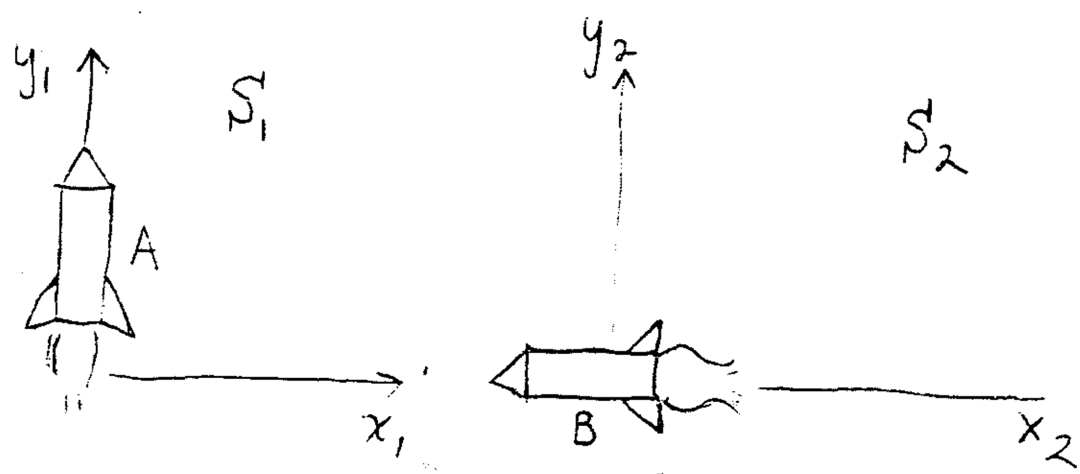
$$x_2 = \gamma(x_1 - vt_1) \qquad \gamma = (1-\frac{v^2}{c^2})^{-1/2}$$

$$y_2 = y_1$$

$$z_2 = z_1$$

$$t_2 = \gamma(t_1 - \frac{v}{c^2}x_1)$$

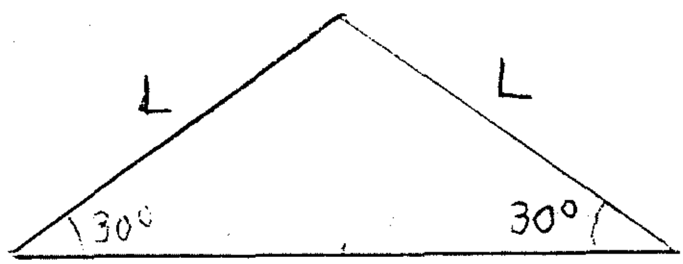
derive the components of rocket A's motion in the frame S_2 and the angle its motion would appear to make with the x_2 axis. What is the speed of rocket A as seen by rocket B?



IV A: Continued

2. A frame of reference S_2 is moving to the right at a speed v with respect to another frame S_1. The following triangular object is at rest in S_2:

Compute the speed v required so that an observer in S_1 would see this object as an <u>isosceles</u> triangle.



IV. B: Thermal Physics

1. Consider 1×10^{22} atoms of ^4He at an initial volume of 10^3 cm^3 at T = 300 K. Given: $k = 1.38 \times 10^{-23}$ J/K; $N_A = 6 \times 10^{23}$ atoms/mole; R = 8.31 J/K mole.

(a) Let the gas expand slowly at constant temperature until the final volume is 2 liters. Find the final pressure and the change in entropy in the expansion.

(b) Consider the system in the same initial state as in part (a). Assume that the expansion is sudden (free expansion). Find the final temperature and the change in entropy in the expansion.

2. Consider a gas of photons (electromagnetic waves) inside a cavity. The energy of the state (neglecting any zero-point energy) having n quanta of the mode having angular frequency ω is $\epsilon_n = n\hbar\omega$.

(a) Show that the partition function Z for the gas is given by

$$Z = \frac{1}{1 - \exp(-\hbar\omega/kT)}$$

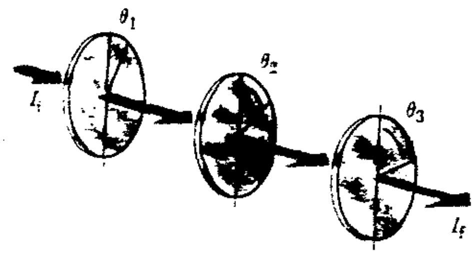
where $\hbar$ and k are Planck's and Boltzmann's constants.

(b) Show that the average thermal energy in the mode is

$$\langle\epsilon_n\rangle = \langle n\rangle\hbar\omega = \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} \quad .$$

IV. C. Optics

1. Three polarizing disks whose planes are parallel are centered on a common axis. The direction of the transmission axis in each case is oriented at an angle θ. Consider the case where $\theta_1 = 0$, $\theta_3 = \pi/2$ and the middle disk rotates about the common axis such that $\theta_2 = \omega t$. Unpolarized light of intensity I_i is incident on disk 1 from the left.



(a) What is the intensity of the light transmitted by disk 1?

(b) With what frequency will I_f, the intensity of the light emerging from disk 3, be modulated?

(c) Find an expression for I_f/I_i as a function of time.

2. Two positive thin lenses with focal lengths of 0.30 m and 0.50 m are separated by a distance of 0.20 m. A small frog rests on the common axis of the two lenses 0.50 m in front of the first lens. Locate the resulting image with respect to the second lens. State whether the image is real or virtual, upright or inverted, magnified or reduced.

IV. D: Astrophysics

1. (a) What is the annual parallax shift displayed by a star at a distance of 150 parsecs? If a typical error is ± 0.005, how many observations will be needed to get the distance with a 10% accuracy?

Suppose we could set up an observatory on Mars to measure stellar parallaxes. Would there be an advantage over earth based observations? If yes then how much?

(b) What is the absolute magnitude and corresponding luminosity (in solar units) of a star if its visual magnitude is 13.7 and its distance is 150 parsecs?

If the star emits most of its light at 10000 Angstroms, what is its approximate temperature?

(c) The same star has a flux (visible light) of $F_{*} = 1.39 \times 10^{6}$ ergs cm^{-2} sec^{-1} and a distance of 50 parsecs. What is its approximate diameter? What angle will it subtend on the sky?

Some useful numbers: $\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} sec^{-1} K^{-4};

1 parsec $= 3.086 \times 10^{18}$ cm;

$\sigma_w = 0.29$ cmK . 1 Angstrom $= 10^{-8}$ cm.

The apparent magnitude of the sun is -27.

The luminosity of the sun is 3.9×10^{33} erg sec^{-1}.

The radius of the sun is 6.96×10^{10} cm.

IV. D: Continued

2. (a) Derive Kepler's Third law as a relationship between mass, orbital radius and period that will be useful for calculating the masses of celestial objects. Assume a circular orbit in your derivation. First obtain the formula for mass in kg then the form for mass in solar mass units with years and AU as time and length units.

(b) Derive the Virial Theorem (the general relation between kinetic and potential energy for two point masses held together by (inverse square law) gravitational forces in a circular orbit.

(c) Use the relation from (a) to determine the mass of the Sirius AB binary system in solar masses (a = semimajor axis = 7.62 arcsec; d = distance = 2.65 parsecs; P = orbital period = 49.9 years).

(d) Determine the individual masses for the binary components. Component B is 2.3x farther from the center of mass than A.

(e) Sirius B has an apparent magnitude of about +9 and effective temperature T= 29500K. What kind of star is this companion? Be specific about size and density, i.e., do calculations.

Possible useful constants besides those in the previous problem:

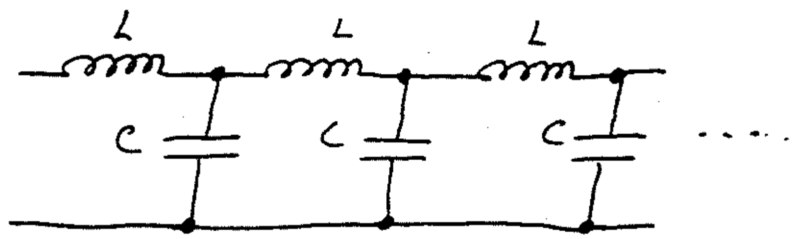
One solar mass = 1.989×10^{30} kg

1 AU = 1.496×10^{11} m

1 parsec = 206265 AU

IV. E. Electronics

1. Calculate the impedance of this infinite chain of inductances and capacitors. Briefly, describe the output of this circuit for DC and AC inputs.



2. Describe the output of this circuit over three cycles.