Qualifying Examination — January 2008

General Instructions

• No reference materials are allowed (except for the use of a calculator).

• Do all your work in the corresponding answer booklet.

• On the cover of each answer booklet put only your *assigned number* and the part number/subject.

• Turn in the questions for each part with the answer booklet.

• 90 minutes are allotted for each part, except for Thermal Physics (45 minutes).

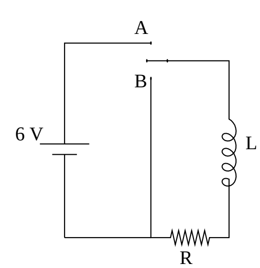
Part I: Electricity and Magnetism

Do any 5 of the 6 problems

1.) A small piece of a circuit carrying current 3mA in the z direction is $\Delta\ell$=10^{-4}m long. Its center is at the point $(x, y, z) = (0.05, 0.05, 0.05)$m. What is the approximate $\vec{B}$ field at the point $(x, y, z) = (0.02, 0.03, 0.04)$m due to this small element of current? [$\mu_o/(4\pi)$=10^{-7} Tm/A.]

2.) A circular wire loop of resistance R lies in the x,y plane and a uniform $\vec{B}$ field points in the z direction. The loop is then manipulated so that its area oscillates with time as $A = A_0 \sin(\omega t)$. What is the induced current in magnitude and direction?

3.) In the circuit shown below a $6V$ battery is connected through a double throw switch to an inductance of $6mH$ and a resistance of 3000Ω. Initially the switch is open and the current is zero. At time t=0 the switch is thrown to position A. At time $t = t_1 = 1\mu s$ the switch is then thrown to position B. What current is flowing in the inductance at time $t = t_2 = 2\mu s$?



4.) The scalar potential on a spherical shell of radius R is

$$V(R, \theta) = V_0 \cos\theta.$$

a) What are the electric fields inside and outside the shell?
b) What is the charge density on the shell?
(*Hint:* In a charge-free region the potential can be written as a sum of terms of the form $(a_l(r/R)^l + b_l(r/R)^{-l-1})P_l(\cos\theta)$.)

5.) A line segment from $a/2$ to $3a/2$ on the z-axis carries a uniform charge per unit length λ. An identical line charge lies from $a/2$ to $3a/2$ on the x-axis.
What is the potential at the origin?

6.) In a charge free region of space there is an oscillating magnetic field

$$\vec{B} = B_0 \hat{\zeta} \left(e^{i(\vec{k}_1 \cdot \vec{r} - \omega t)} - 2e^{i(\vec{k}_2 \cdot \vec{r} - \omega t)} \right) \quad .$$

What is the $\vec{E}$ field and what is the real part of the Poynting vector?

Part II: Quantum Mechanics

Do any 5 of the 6 problems

1. Suppose that the initial state of a particle in a harmonic oscillator potential is

$$\psi(x,0) = \frac{1}{\sqrt{3}}\psi_0 + \sqrt{\frac{2}{3}}\psi_1 \,.$$

where ψ_0 and ψ_1 are the ground and first excited states, respectively, and are given as follows:

$$\psi_0 = (\frac{1}{\sqrt{\pi}})^{\frac{1}{2}} e^{-q^2/2}$$

$$\psi_1 = (\frac{2}{\sqrt{\pi}})^{\frac{1}{2}} q e^{-q^2/2}$$

$$q = \sqrt{\frac{\omega m}{\hbar}} x$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{\sqrt{\pi}}{a}$$

At some later time $t > 0$ find: a) the standard deviation Δx, b) the standard deviation Δp, as functions of time.

2. Consider the one-dimensional wave function

$$\psi(x,t) = A\, \mathrm{e}^{-\lambda\, |x|}\, \mathrm{e}^{-i\omega t}$$

a) Normalize $\psi(x,t)$

b) Determine the expectation values of x and x^2.

c) Find the standard deviation, σ, of x. Sketch the graph of $|\psi|^2$ and mark the points $(<x> +\sigma)$ and $(<x> -\sigma)$. What is the probability that the particle will be found outside of this range?

3. The electron in a hydrogen atom is in the state

$$|\psi(t)>= \frac{1}{\sqrt{3}} e^{-iE_1 t/\hbar}|E_1, 0, 0> - \sqrt{\frac{2}{3}} e^{-iE_2 t/\hbar}|E_2, 1, 1> ,$$

where the $|E_n, l, m>$'s are eigenvectors of the Hamiltonian, the angular momentum operator squared, and the z-component of the angular momentum operator.

a) Find the expectation value of H. Express your answer in terms of electron volts (eV).

b) Find the expectation value of L^2.

4. Consider a system in which there are just two linearly independent states. The Hamiltonian for this system is

$$\mathbf{H} = \omega \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ,$$

where ω is a constant.

(a) Determine the eigenvalues of this Hamiltonian.

(b) Find the normalized eigenvectors for this Hamiltonian.

(c) Suppose the system is in the state $\frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the probability of measuring the energy eigenvalues found in (a).

5. Suppose that the electron in a hydrogen atom is perturbed by a repulsive potential concentrated at the origin. Assume the potential has the form of a 3-dimensional delta function, such that the perturbed Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r} + A\delta^3(\mathbf{r})$$

where A is a constant.

a) To first order in A, find the change in the energy of the state with quantum numbers $n = 1, l = 0$. [Hint: $\psi_{n00}(0) = 2/[(4\pi)^{1/2}(na_0)^{3/2}]$, where a_0 is the first Bohr radius $a_0 = \hbar^2/\mu e^2$, $\mu =$ reduced mass.]

b) Find the change in the wave function.

6. A particle of mass m is confined to a one-dimensional box of width L, that is, the potential energy of the particle is infinite everywhere except in the interval $0 < x < L$, where its potential is zero. The particle is in its ground state. What is the probability that a measurement of the particle's position will yield a result in the left quarter of the box?

Part III: Classical Mechanics

Do any 5 of the 6 problems

1. (a) Assume that the force of air resistance acting on a falling object is proportional to the speed of the object: $\mathbf{F} = -b\mathbf{v}$. Solve Newton's second law to find an expression for the velocity of the falling object, assuming it starts from rest.

(b) Suppose that the force of air resistance acting on a falling coffee filter is proportional to the square of the speed: $\mathbf{F} = -c|\mathbf{v}|\mathbf{v}$. Suppose in addition that the filter attains its terminal velocity almost instantly after being dropped. If a filter of mass m takes one second to fall one meter, from what height should a filter of mass $2m$ be dropped in order to also fall in exactly one second?

2. A cubical block of mass m slides without friction down the hypotenuse of a triangular block of mass M. The triangular block slides without friction on a horizontal surface.
(a) What quantity or quantities are conserved in this system?
(b) Find the speed of the triangular block, as a function of m, M, and θ, when the mass m reaches the bottom of the incline. Assume the center of mass of m starts from rest at a height H above the horizontal surface. Ignore the size of the block.

3. Masses m_1 and m_2 are joined by a massless string of length L. The string passes over a solid ideal (rotates without slippage) cylindrical pulley of mass M and radius R, with m_1 hanging freely and m_2 sliding on a frictionless incline of angle θ. The incline is fixed in place. The moment of inertia of the pulley is $I = (1/2)MR^2$.
(a) Write down the Lagrangian for this system.
(b) Find the equation of motion and solve it to find the acceleration of the masses.

4. Two identical balls of mass 0.8 kg hang side by side from strings of length L. The balls are made of clay and should be thought of as simple pendulums. One ball is pulled to the side until it is a height of 0.5m above the other ball, and it is then released. The two balls collide and stick together. To what height do they rise on the other side?

5. The effective potential for the radial motion of a test particle in the presence of a black hole is

$$V_{eff}(r) = -\frac{\alpha}{r} + \frac{\beta}{r^3} + \frac{\ell^2}{2mr^2}, \qquad \alpha > 0, \beta = -\frac{2\alpha l^2}{m^2}$$

For what range of the angular momentum ℓ are there stable circular orbits and at what radii r do they occur? What is the smallest circular orbit?

6. A rectangular block of wood of height h and area A oscillates vertically in water. Find the period of small oscillations about the equilibrium position of the block, in terms of its density ρ, the density of water ρ_w, and the dimensions of the block.

Part IV: Thermal Physics

Do any 2 of the 3 problems

1. A mole of an ideal gas expands according to the law $PV^2 = \text{const.}$
(a) Determine whether its temperature goes up or down.
(b) What is the heat capacity of the gas during this process in terms of the gas constant R?

2. A box is separated by a partition which divides its volume in the ratio 3:1. The larger portion of the box contains 1000 atoms of Ne gas; the smaller, 100 atoms of He gas. A small hole is punctured in the partition, and one waits until equilibrium is attained.
(a) Find the mean number of atoms of each type on either side of the partition.
(b) What is the probability of finding 1000 atoms of Ne gas in the larger partition and 100 atoms of the He gas in the smaller. (i.e., the same distribution as in the initial system)?

3. One kilogram of water at 0°C is brought into contact with a large heat reservoir at 100°C. When the water has reached 100°C, what has been the change in entropy of (a) the water, (b) the heat reservoir, and (c) the entire system consisting of both water and heat reservoir? (Assume c_p=4.186$\times 10^3$ J(kg K)$^{-1}$.)