QUALIFYING EXAMINATION

JANUARY 1998

General Instructions: No reference materials (except for the use of a calculator) are permitted. Do all your work in the answer booklet. Turn in the questions for each part with the answer booklet. There are 90 minutes allotted for each part, with a 30-minute break in between.

PART I: Classical Mechanics

Work any 5 of the 6 problems.

1. A floor mop of mass M is pushed with a force F directed along the handle, which makes an angle θ with the vertical. The coefficient of kinetic friction with the floor is μ. Find the force required to slide the mop with constant velocity.

2. A body of mass m with initial velocity v_0 is subjected to a frictional force $F = -b\sqrt{v}$, where b is a constant. For the one-dimensional case, set up and solve the equation of motion for $v(t)$ and $x(t)$. Assume $x(0) = 0$, and that the initial velocity is in the $+x$ direction.

3. A thin circular hoop of mass M and radius R rolls down an inclined plane without slipping. Find the speed of its center of mass when the hoop reaches the bottom. The plane is inclined at an angle θ with the horizontal, and the plane has length L.

4. Determine which of the following forces are conservative, and determine the potential energy function for those that are:

a) $\mathbf{F} = ax\hat{\mathbf{i}} + by\hat{\mathbf{j}} + cz\hat{\mathbf{k}}$

b) $\mathbf{F} = ax^2yz\hat{\mathbf{i}} + axy^2x\hat{\mathbf{j}} + bxyz^2\hat{\mathbf{k}}$

5. Two people are holding the ends of a uniform plank of length L and mass M. One of them suddenly lets go. Find the initial acceleration of the free end.

6. Use Lagrange's equations to find the differential equations of motion for a simple pendulum.

PART II: Electricity and Magnetism

Do any 5 of the 6 problems.

1. An object with spherical symmetry has charge density $\rho(r) = Ar$ for $r < R$ and $\rho(r) = 0$ for $r \geq R$ (i.e. it has radius R). Thus the charge density is highest near the surface.

a) compute the electrostatic potential and electrostatic field everywhere.

b) An electron with opposite charge $-e$ and mass m starts out at $r = \infty$ and falls toward the sphere. What is the velocity v as a function of r? Assume that the electron interacts only electrostatically with the sphere, as though confined to a narrow tunnel.

c) if the electron starts at $r = R$, what is its period of oscillation?

2. A spherical surface has electrostatic potential

$$V(R, \theta) = V_0 + V_1 \cos\theta + V_2 \cos^2\theta,$$

where θ is the polar angle. What is the potential $V(r, \theta)$ at any point outside $(r > R)$? Assume there is no charge outside the surface.

3. A long wire carrying current I points along the z axis direction at $x = y = 0$. The part from $z = -d/2$ to $z = +d/2$ is cut out, and the cut ends are connected to circular plates of radius $a >> d$ and centered at $x = y = 0$, in the planes $z = \pm d/2$.

a) A circular Amperian loop of radius $R >> a$ is drawn around the wire, in a plane $z = B >> a$ far from the capacitor plates at $z = \pm d/2$. Does the line integral of B around the loop change as the loop is moved to $z = 0$? If so, how does it change?

b) Write Ampere's Law for both positions of the loop.

c) Use the result of (b) to relate the electric field E between the plates to the current I.

4. A long line charge has linear charge density λ (coulombs/meter or esu/cm).

a) compute the electric field at a distance r from the charged line. Indicate its direction in a sketch.

b) Now suppose the wire moves along itself at velocity v, so that there is an electric current $I = \lambda v$. Compute the magnetic field B at a distance r. Show its direction in the sketch.

c) Compute the Poynting vector (energy flux density). Assume that the E field is unaffected by the motion.

d) Compute the total energy flux through an annulus of inner and outer radii a and b, in a plane perpendicular to the line charge. What is the energy flux through the entire plane?

5. a) Write Maxwell's equations.

b) Apply them to a plane wave

$$E(r,t) = R_e E_0 e^{(ikr - i\omega t)}$$

$$B(r,t) = R_e B_0 e^{(ikr - i\omega t)}$$

to relate E_0 to B_0. Sketch k, E_0, and B_0.

6. a) In a magnetic material with magnetization density M, write a formula for the magnetization current J_M.

b) Write Ampere's Law for the case $E = 0$, relating curl B to the total current $J_{free} + J_M$.

c) Write a formula (in terms of M and B) for a field H whose curl involves J_{free} but not J_M.

d) For an isolated (i.e. $B \to 0$ as $r \to \infty$) uniformly magnetized sphere, make two sketches showing the lines of B and H.

PART III: Quantum Mechanics

Vectors are indicated by **bold** print.

1. Assume that the states in a multi-electron atom are specified by the principal quantum number n, the angular momentum l, the z-component of l l_z, and the z-component of the spin s_z.

a) For a given value of n, how many electrons can have the same value of l?

b) How many electrons can have the same value of n? Show how you arrive at your answers.

2. A particle of mass m and energy $E > V_0$ is incident on a one-dimensional barrier $V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$. Define the transmission coefficient and derive an expression for it in terms of E and V_0.

3. The ground state of a particle of mass m in a one-dimensional simple harmonic oscillator potential ($V(x) = m\omega^2 x^2/2$) has the form

$$\Psi_0 = ae^{(-\alpha^2 x^2)}.$$

Determine the value of the constant α and the energy of the state.

4. Using the commutation relations for the angular momentum operator, show that a state ψ which is an eigenstate of L_z cannot simultaneously be an eigenstate of L_x or L_y (except for the case where $\mathbf{L}^2\psi = 0$). Show that it is possible for a state to be a simultaneous eigenstate of $\mathbf{L}^2$ and L_z by proving that $\mathbf{L}^2$ commutes with any component of $\mathbf{L}$. It is only necessary to prove this for one component of $\mathbf{L}$.

5. The ground state of the hydrogen atom is given by

$$\Psi(r) = Ae^{(-r/a_0)}$$

where a_0 is the Bohr radius. Determine the constant A in terms of a_0 if the wavefunction is normalized to unity. For what value of r is the electron most likely to be found?

6. The energy eigenstates (normalized and orthogonal) of a system are given by

$$H\psi_n(\mathbf{r}) = E_n\psi_n(\mathbf{r}).$$

If the (normalized) wavefunction of the system at $t = 0$ is given by

$$\psi(\mathbf{r}, 0) - \psi_1(\mathbf{r})/2 + c\psi_3(\mathbf{r})$$

with $c > 0$, determine c and the expectation value of the energy. Write down the wavefunction at time $t > 0$.

PART IV: Mixed Topics

Do problems from 4 of the 5 sections (choose either the astrophysics or electronics section). Do 6 of the 8 problems in these sections, with at least one from each of the 4 sections. Use a different answer book for each lettered section.

a: Relativity

1. Two neutrons with opposing initial velocities of $0.8c$ make a head-on collision.

a) What is the total energy of the system in units of the neutron rest mass m_n and c?

b) What is the collision speed in the frame of reference of one of the colliding particles, in units of c?

c) What is the effective mass of the neutrons in the hypothetical observer's reference frame, in units of m_n?

2. An observer in frame S_2 holds a meter stick at an angle of 45° from the relative motion between frames S_2 and S_1. If S_2 is moving at speed $0.98c$ with respect to S_1, what is the length and angle of the meter stick as measured by an observer in S_1?

B. Thermodynamics

1. Consider a column of atoms (each of mass M) of an ideal gas at temperature T in a uniform gravitational field g. Find the thermal average kinetic energy per atom. Take the zero of gravitational energy at the bottom (height $h = 0$). Integrate from $h = 0$ to infinity.

2. Consider a system of N distinguishable particles. Each particle can be in two states, one of energy 0 and the other of energy ϵ. The particles are in thermal equilibrium at a temperature T.

a) What fraction of the particles have energy 0?

b) Find an expression for the specific heat of the system.

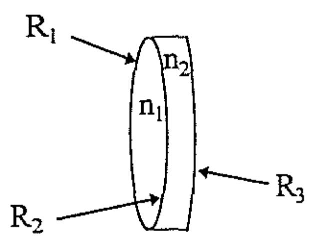
C. Optics

1. Consider the compound lens shown below. The indices of refraction of lenses 1 and 2 are $n_1 = 1.5$ and $n_2 = 2.0$. The *magnitudes* of the radii of curvature of the surfaces are $|R_1| = |R_2| = 20$ cm and $|R_3| = 40$ cm.

a) Calculate the focal length of the compound lens, assuming the thin-lens approximation.

b) An object is placed 20 cn to the left of the lens. Calculate the position of the image, the lateral magnification, and the longitudinal magnification.

c) Draw a ray diagram to locate the image, using the three principal rays.



2. The intensity distribution of single-slit diffraction in the Fraunhofer limit is given by

$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2}$$

where $a = \frac{\pi a \sin\theta}{\lambda}$, a is the slit width, and θ is the diffraction angle.

a) Derive this expression.

b) Plot I versus $\sin\theta$ and label the abscissa in units of λ/a.

c) Calculate the intensity of the first secondary maximum relative to the intensity of the central maximum.

D. Astrophysics

1. Consider a spherical distribution of stars which is initially "cold" (i.e. the stars are motionless) and extends to a radius R_i. The stellar distribution then collapses, due to self-gravity, and virializes (i.e. $2T = -W$, or twice the total kinetic energy equals the (negative of the) potential energy.)

a) What is the collapse factor of the virialized system ? That is, how much smaller is the final system, which has radius R_f, compared to the initial size R_i ? So that you can neglect some geometric coefficients which depend on the detailed spatial distribution of stars, assume the initial and final stellar distributions have the same shape density distribution.

b) Show that a body of mass m in a circular orbit around a much more massive body with mass M, with separation R, is "virialized" (that is, $2T = -W$)

2. The Sun's luminosity is 4×10^{33} erg/s. The Earth is 1.5×10^{13} cm away from the Sun (on average) and has a radius of 6.4×10^{8} cm.

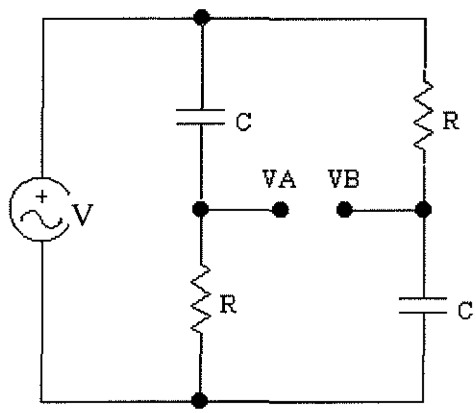
a) The Earth reflects 40% of the light from the Sun, so how much energy is absorbed by the Earth per second ?

b) The energy produced by nuclear warheads is usually measured in kilo- or megatons of TNT. A kiloton of TNT corresponds to 4×10^{19} erg. What is the equivalent kilo- or megatonnage of solar energy absorbed by the Earth each second ?

c) There are nearly 6 billion people on the Earth, each of whom is radiating black body radiation with a characteristic temperature of ≈ 300 degrees Kelvin. What is the rate of energy lost by all people, due to their being warm bodies ? The Stephan-Boltzmann constant is $\sigma = 5.7 \times 10{-}5$ erg/s/cm^2/deg^4.

E. Electronics

1. The circuit below is a phase shifter. Find the magnitude and phase of $VA - VB$ with respect to the input voltage $V = V_0 \cos(\omega t)$.



2. Find the resistance between points A and B in the circuit below.