QUALIFYING EXAMINATION

JANUARY 1997

General Instructions: No reference materials (other than a calculator) are permitted. Do all work in your answer booklet. Turn in the questions for each part with the answer booklet. You may finish Part I early, turn it in and start on Part II.

PART I: Classical Mechanics

Work any 5 of the 6 problems.

1. After the engine of a motorboat having a mass m and traveling at a speed v_0 is shut off, there is a frictional drag proportional to the square of the velocity; i.e.,

$$F(v) = -kv^2 \quad .$$

   a) Write down the equation of motion and find v (t).

   b) Find the motion x (t).

   c) Show that the speed after traveling a distance x is given by

$$v = v_0\, e^{-kx/m} \quad .$$

2. A particle of mass m moves under the action of a central force whose potential is

$$V(r) = Kr^4 ,\ K > 0 \quad .$$

   a) Find the energy and angular momentum such that the orbit will be a circle of radius r_m about the origin.

   b) Find the period of the circular motion.

3. A coal train of mass m_0 initially contains an equal mass of coal m_0. At t = 0, a constant external force F is applied. As the train begins to move, the coal starts dropping out through a hole in the bottom at a constant rate b = - dm/dt, for b > 0.

   a) What is the speed of the train when all the coal has dropped out?

   b) By what factor is the speed found in part a) greater than the speed would have been if no coal had dropped out?

4. a) Suppose a point mass m is dropped from a height h (where $h << R_e$, R_e being the radius of the Earth assumed to be spherical). Show that the speed with which the mass hits the Earth is approximately

$$v = \sqrt{2gh}\,\left(1 - \frac{h}{2R_e}\right) .$$

b) Show that a mass m in free fall in the Earth's gravitational field from infinity to R_e will have exactly the same speed when it hits the surface as another mass m′ falling from a height $h = R_e$, assuming that for m′ the acceleration g is constant.

5. A pendulum bob of mass m is suspended by a massless rod of length L from a point of support. The point of support moves along a horizontal axis according to the equation

$$x = a \cos \omega t .$$

Assume that the pendulum swings only in a vertical plane containing the x axis and let θ be the angle between the string and the vertical.

a) Set up the Lagrangian and write out the corresponding Lagrangian equation(s).

b) Show that if θ is small, one can reduce the problem to that of a forced harmonic oscillator. Find the steady-state solution.

6. Consider a ladder 5 m long with a mass of 25 kg which leans against a smooth (frictionless) wall. The ladder makes an angle of 60° with respect to the rough (not frictionless) ground.

a) Find the force exerted on the ladder by the wall.

b) How high off the ground can a person having a mass of 50 kg climb before the ladder slips along the floor if the coefficient of static friction is 0.5?

PART II: Electricity and Magnetism

Work any 5 of the 6 problems.

1. A sphere of radius R centered at the origin contains a spherically symmetric charge density of the form

$$\rho(r) = \frac{Q}{Rr^2}\, e^{-r/R}$$

What is the electric field as a function of position inside and outside of the sphere?

2. A right angle pie shaped region of radius R and infinite in the z direction contains no charge. What is the electrostatic potential $\Phi(r, \theta)$ in the region $0<r<R$, $0<\theta<\pi/2$ if on the boundary the potentiall satisfies

$$\Phi(r, 0) = \Phi(r, \pi/2) = 0$$

and

$$\Phi(R, \theta) = V \sin(8\theta) \qquad ?$$

3. A grounded conductor occupies the region $x<0$, all z and $y<0$, all z. A charge Q is brought to position $(x, y, z) = (d, d, 0)$ for positive d. What is the electrostatic force on the charge?

4. An object of charge e and mass M falls from rest at height h. It is acted upon by a constant downward force of magnitude mg and by a magnetic force due to a constant B field in the x direction. If the vertical is taken to be the z direction, what is the magnitude of the maximum velocity attainable by the object in the y direction?

5. A circular conducting loop of radius b and resistance R lies in the x-y plane. Starting from time t=0 a $\bar{B}$ field in the z direction decreases exponentially as $\bar{B}(t) = B_0 \hat{k}\, e^{-t/\tau}$.

   a) Does the current flow clockwise or counterclockwise as seen from the region of positive z?

   b) What is the total amount of Joule heat deposited in the ring between $t = 0$ and $t = \infty$?

6. The surface of an otherwise infinite plane grounded conductor has a small hemi-spherical bump of radius b. (Infinitely) Far from the bump there is a uniform electric field perpendicular to the plane, $\bar{E} = E_0 \hat{e}_z$. What is the electric field in the vicinity of the bump?

PART III: Quantum Mechanics

Work any 5 of the 6 problems.

1. The wavefunction for a particle in one dimension is

$$\Psi(x) = \frac{\sqrt{15}}{4}\left(1-x^2\right) \quad , \qquad -1 \le x \le 1$$

$$\Psi(x) = 0 \quad , \qquad |x| > 1$$

Verify that the normalization is correct and compute the probability of finding the particle between $x = -\frac{1}{2}$ and $\frac{1}{2}$.

2. Compute the expectation value for x, x^2, p, p^2 (p being the momentum operator), using the wavefunction in #1 and show that the Heisenberg uncertainty principle holds, $\Delta x \Delta p \geq \hbar/2$.

3. A one gram mass is attached to a spring with spring constant equal to .4 N/m. Quantum mechanics says that its lowest energy is not zero, but rather how much? Give a rough estimate of the uncertainty in measuring the position of the mass when the oscillator has its lowest energy.

4. A particle is in a 3 dimensional square well, $0 \leq x, y, z \leq L$. What is the lowest energy eigenvalue having a six-fold degeneracy and what are the associated eigenfunctions?

5. a) If the differential representations for the y and z components of angular momentum are

$$L_y = -i\hbar \left(\cos\varphi \frac{\partial}{\partial\theta} - \cot\theta \sin\varphi \frac{\partial}{\partial\varphi}\right)$$

$$L_z = -i\hbar \frac{\partial}{\partial\varphi}$$

compute the differential representation for the x component of angular momentum?

b) Show that $\cos^2\theta - \frac{1}{3}$ is an eigenfunction of L_z and $\bar{L}^2$ and find their eigenvalues.

6. At time $t = 0$ a harmonic oscillator is in a state described by the normalized wave function

$$\Psi(x, 0) = \sqrt{\frac{1}{5}}\, U_o(x) + \sqrt{\frac{1}{2}}\, U_1(x) + C_3\, U_2(x)$$

where $U_n(x)$ is the n^{th} orthonormal stationary eigenfunction, $n = 0$ being the ground state.

a) Determine C_3.

b) What is the expectation value of the energy?

c) Write down the wavefunction at time t.

PART IV: Mixed Topics

Do problems from 4 of the 5 sections. Astrophysics may be chosen only in place of electronics. Do 6 of the 8 problems with at least one from each of the 4 chosen sections. Use a different answer book for each lettered section.

A: Relativity

1. A hypothetical particle having a mass of 6×10^{-30} kg decays at rest into an electron and a positron. (The positron, being the antiparticle of the electron, has the same mass as the electron.) Calculate

   a) The total energy of the electron.

   (b) The magnitude of its momentum.

   (c) Its speed.

2. In one inertial reference frame, label it A, two events are observed to occur in the x-y plane, the first event at $x=1000$ m, $y=500$ m, and $t=10^{-6}$ s and the second event at $x=500$ m, $y = -500$ m, and $t = 2\times10^{-6}$ s. The same two events are observed from another inertial frame, label it B, which moves at a velocity of $-0.8c$ along the x-axis as seen from frame A. As observed in B, find

   a) The spatial distance between the two events.

   b) Their separation in time.

B: Thermodynamics

1. A moveable wall divides a closed heat-insulated cylinder into two compartments, each containing one mole of an ideal gas. Starting from an equilibrium with the wall clamped to maintain twice as high a pressure from one to the other side, the wall is then released to move freely until a new equilibrium is reached. Find the change in entropy.

2. Two identical bodies of constant heat capacity C are used as heat reservoirs for an unspecified heat engine E, operating between them in a series of repeated cycles. Given the initial absolute temperatures T_{1i} and T_{2i} of the bodies, find the maximum amount of work delivered by E.

C: Optics

1. Derive the law of reflection and the law of refraction using ONE of the following: Huygens construction, Fermat's principal or electromagnetic theory.

2. An object 2 cm tall is located 10 cm to the left of a thin converging lens having a focal length of 4 cm. A second converging lens, having a focal length of 5 cm, is located 3 cm to the right of the first lens. Use simple ray tracing to find the location and height of the final image. Use the paraxial thin lens formulas to verify your results for the position and size of the image. Two copies of an accurate figure of the object and lenses have been provided; do the ray tracing part of this question on one of them.

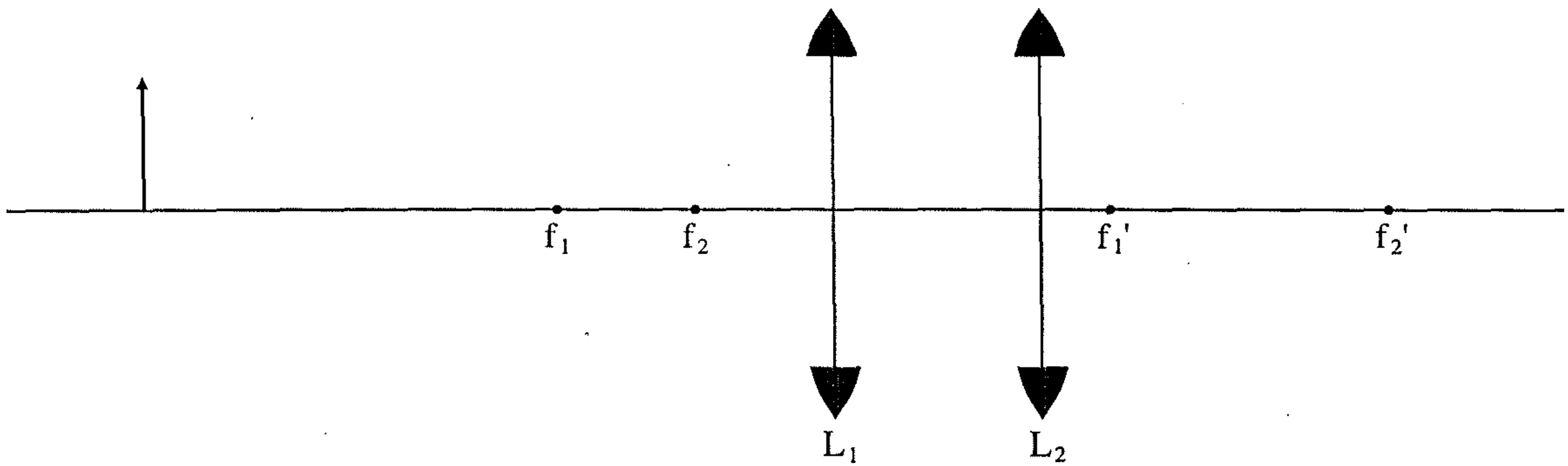

Diagram for Optics Question 2

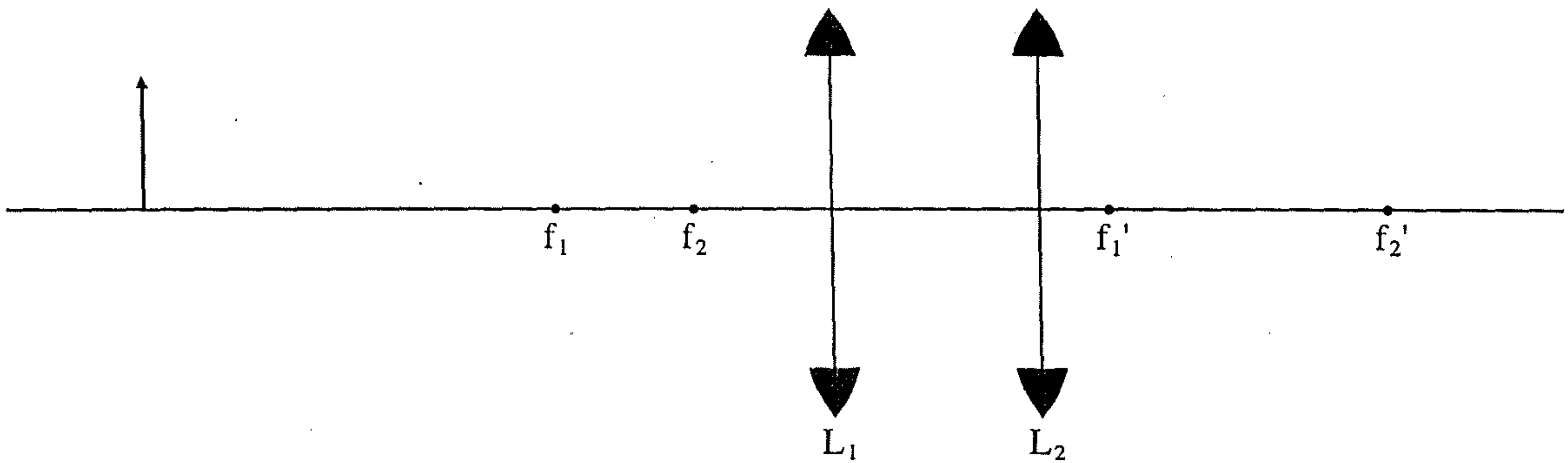

Diagram for Optics Question 2