UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Illinois Center for Advanced Studies of the Universe

New developments of spin hydrodynamics

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High Energy Theory Seminar - The University of Alabama

March 19, 2021

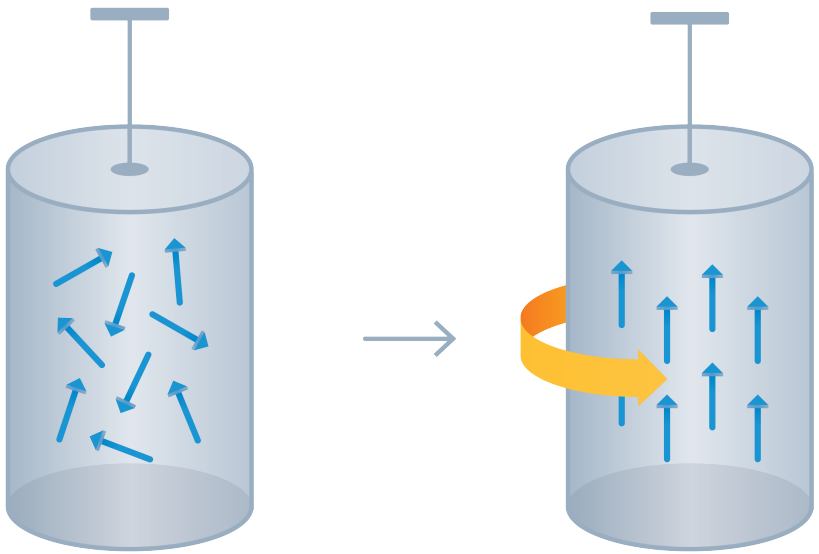
Vorticity and spin physics in heavy-ion collisions

Vorticity

$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{v}$$

Rotation and polarization

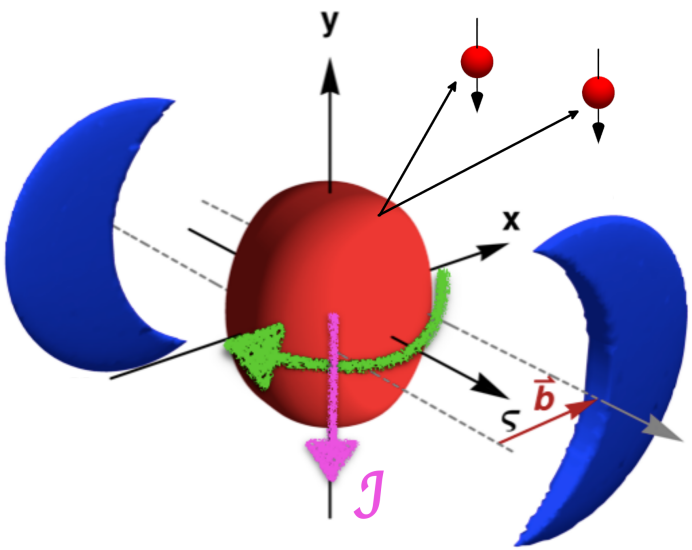
- Condensed matter: **Barnett effect**



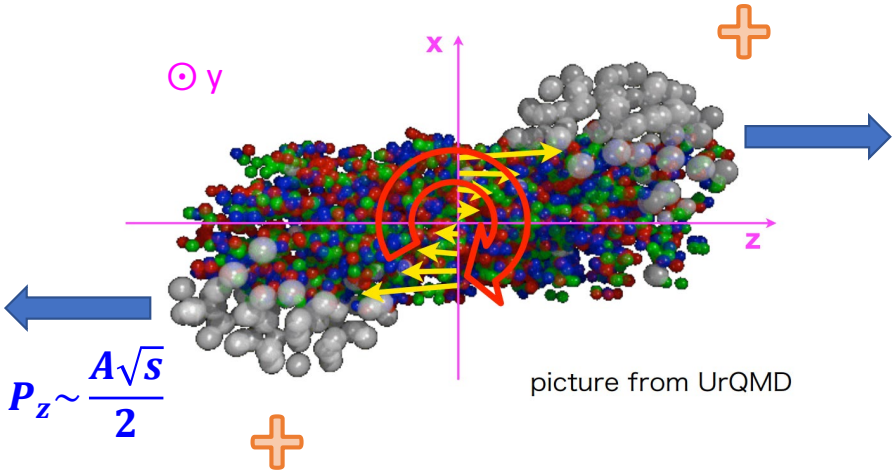
Ferromagnet gets magnetized when it rotates

Polarization effects through rotation in heavy-ion collisions? Yes!

Noncentral heavy-ion collisions



picture from Florkowski, Ryblewski, Kumar, Prog. Part. Nucl. Phys. **108**, 103709 (2019)



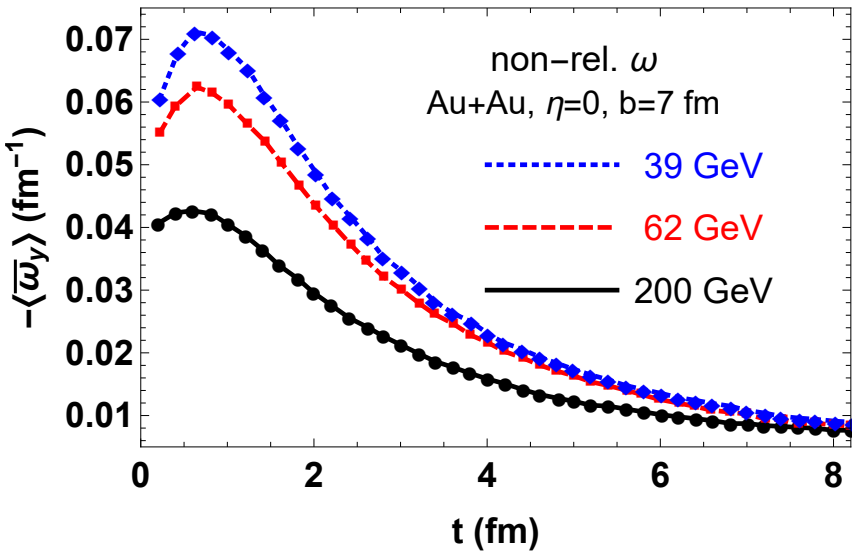
Large global angular momentum

$$J \sim \frac{A\sqrt{s}}{2} b \sim 10^5 \hbar$$

$\Rightarrow$ Vorticity of hot and dense matter $\Rightarrow$ particle polarization along vorticity

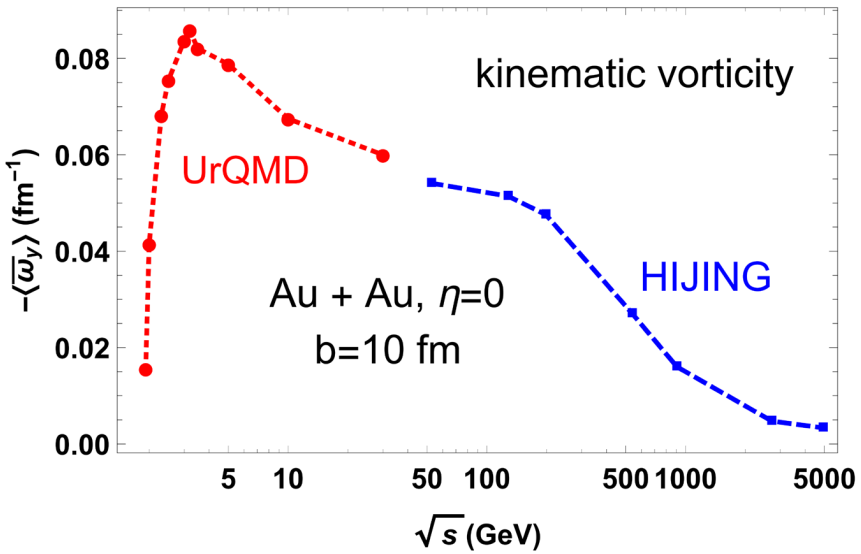
Vorticity in heavy-ion collisions

Time dependence



Jiang, Lin, Liao, PRC 94, 044910

Energy dependence



Deng, Huang, Ma, Zhang, PRC 101 064908
Deng, Huang, PRC 93 064907

see also, e.g., Huang, Liao, Wang, Xia, 2010.08937; Huang, 2002.07549; Becattini et al EPJC 75, 406; Csernai, Magas, Wang, PRC 87, 034906; Csernai, Wang, Bleicher, Stoecker, PRC 90, 021904; Ivanonv, Soldatov, PRC 95 054915

- Vorticity decreases at high energies
- Extremely high vorticity: $\omega_y \sim 10^{-2}\,\mathrm{fm}^{-1} \sim 10^{21}\,s^{-1}$

Spin-vorticity coupling

$$\text{Effective interaction} \sim -\vec{S} \cdot \vec{\omega}$$

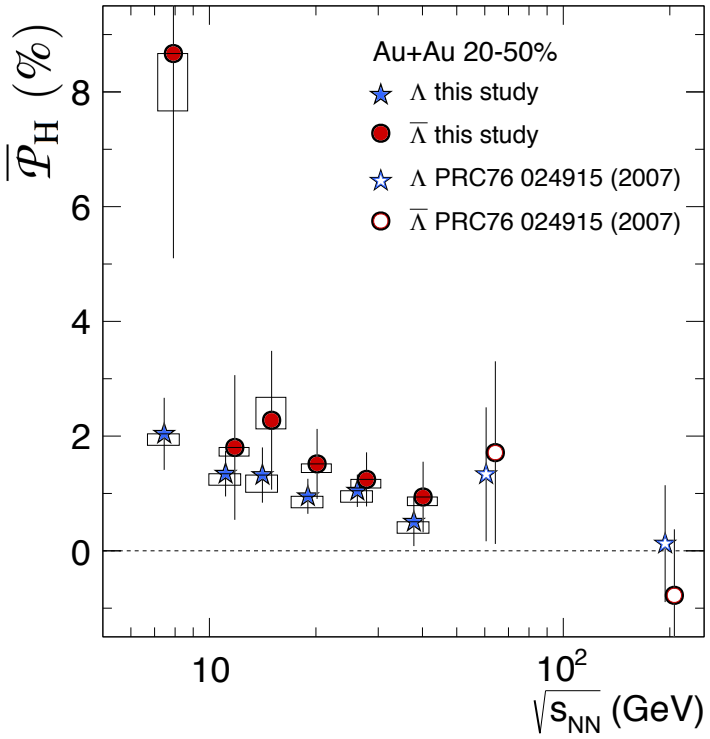
$$\sim \text{Quantum} \cdot \text{Classical}$$

$\vec{S}$ - Particle spin, $\vec{\omega}$ - Medium rotation

- Massive particles $\Longrightarrow$ Λ-baryon polarization
  Voloshin, 0410089; Liang, Wang, PRL **94**, 102301; Betz, Gyulassy, Torrieri PRC **76**, 044901; Becattini, Piccinini, Rizzo, PRC **77**, 024906

Experimental observation - Global Λ polarization

- Polarization along global angular momentum



L. Adamczyk et al. (STAR), Nature 548 62-65 (2017)

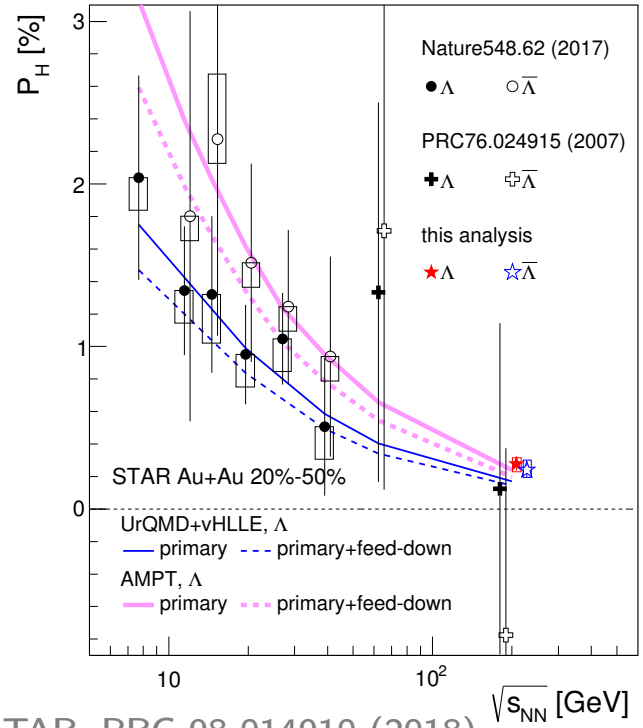
- Weak decay: $\Lambda \to p + \pi^-$ angular distr.: $dN/d\cos\theta = \frac{1}{2}(1 + \alpha|\vec{\mathcal{P}}_H|\cos\theta)$
- Quark-gluon plasma is the "most vortical fluid ever observed"

$$\omega = (P_\Lambda + P_{\bar{\Lambda}})k_B T/\hbar \sim 10^{21}\,\mathrm{s}^{-1}$$

Great Red Spot of Jupiter $10^{-4}\,\mathrm{s}^{-1}$

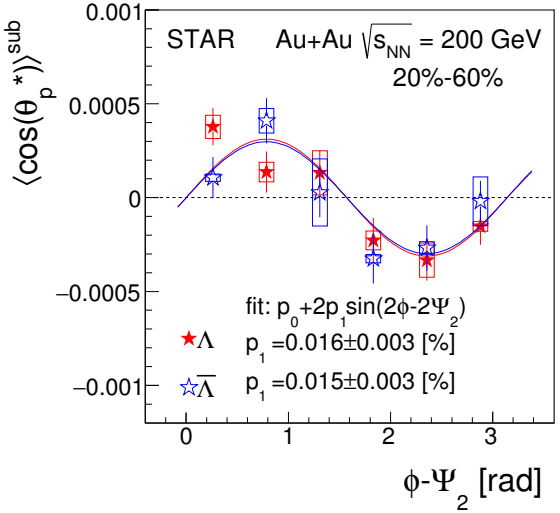
Experiments vs theory: Λ polarization

Global - along J



STAR, PRC 98 014910 (2018)

Longitudinal - along beam axis

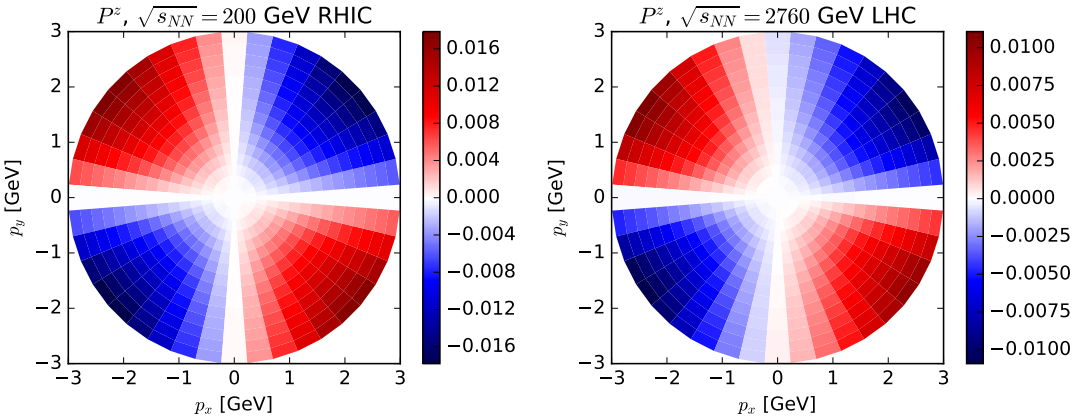


J. Adam *et al.* [STAR Collaboration], PRL 123, 132301 (2019)

$$\Pi^\mu(x,p) \propto (1-n_F)\epsilon^{\mu\nu\rho\tau} p_\nu \varpi_{\rho\tau}$$

$$\varpi_{\rho\tau} = -\frac{1}{2}(\partial_\rho\beta_\tau - \partial_\tau\beta_\rho)$$

Becattini et al An. Phys. (2013)



F. Becattini, I Karpenko, PRL 120, 012302

- Theory assumes local equilibrium of spin degrees of freedom
- "Sign problem" between theory and experiments for longitudinal polarization!

Does spin play a dynamical role in hydro?

- Relativistic hydrodynamics is a good effective theory: $\partial_\mu T^{\mu\nu} = 0$ Heinz, Snellings, Ann. Rev. Nucl. Part. Sci. **63**, 123-151 (2013)

Goal: Relativistic hydrodynamics (classical) with spin (quantum) as dynamical variable

Florkowski, Friman, Jaiswal, ES, PRC **97**, no. 4, 041901 (2018)
Florkowski, Friman, Jaiswal, Ryblewski, ES, PRD **97**, no. 11, 116017 (2018)
Florkowski, Becattini, ES, Acta Phys. Polon. B **49**, 1409 (2018)
Becattini, Florkowski, ES, PLB **789**, 419 (2019)
Bhadury, Forkowski, Jaiswal, Kumar, Ryblewski, 2002.03937, 2008.10976 (2020)
Shi, Gale, Jeong, 2008.08618 (2020)

Starting point: Kinetic theory from quantum field theory

Weickgenannt, Sheng, ES, Wang, Rischke, PRD **100**, no. 5, 056018 (2019)
Weickgenannt, ES, Sheng, Wang, Rischke, 2005.01506 (2020)

- Alternative approaches: Lagrangian formulation, entropy current, ...
  Montenegro, Tinti, Torrieri, PRD 96, 056012 (2017)
  Montenegro, Torrieri, 2004.10195 (2020)
  Hattori, Hongo, Huang, Matsuo, Taya, PLB795, 100 (2019)
  Garbiso, Kaminski, JHEP **12**, 112 (2020)
  Fukushima, Pu, 2010.01608 (2020)
  Li, Stephanov, Yee, 2011.12318 (2020)
  Gallegos, Gürsoy, Yarom, arXiv:2101.04759 (2020)

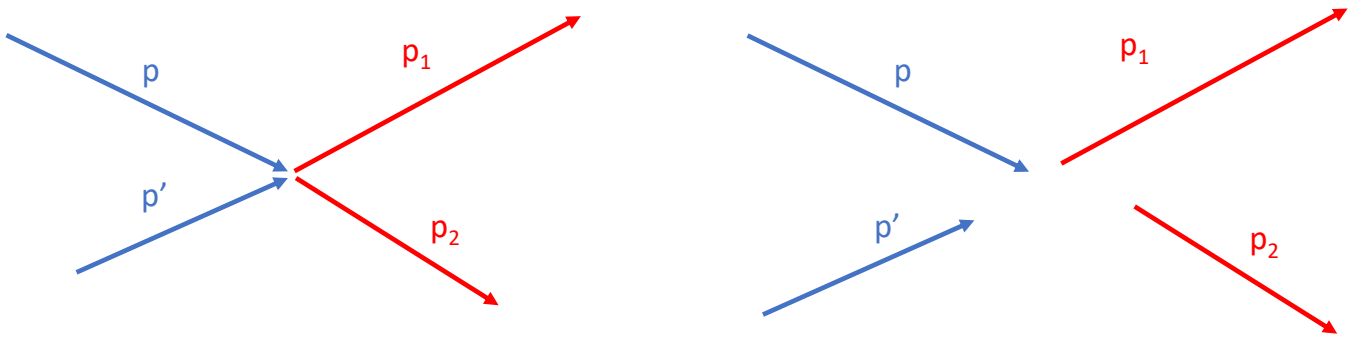
Anticipation of our results

Weickgenannt, ES, Sheng, Wang, Rischke, 2005.01506 (2020), 2103.04896 (2021);

ES, Weickgenannt, 2007.00138 (2020)

- How do we describe the orbital-to-spin angular momentum conversion in kinetic theory?

Nonlocal particle scatterings (finite impact parameter)



Local **Nonlocal**

- And in hydrodynamics?

Antisymmetric part of energy-momentum tensor

Quantum kinetic theory

Classical kinetic theory

- Distribution function

$$f(x,p)$$

- Boltzmann equation

$$p_\mu \partial^\mu f(x,p) = C[f(x,p)]$$

$C[f(x,p)]$ - Collision term

- Hydrodynamics from kinetic theory

$$T^{\mu\nu}(x) = \int d^4p\, p^\mu p^\nu f(x,p)$$

$$\partial_\mu T^{\mu\nu}(x) = 0$$

How to formulate kinetic theory from quantum mechanics?

Wigner function - Quantum mechanics

"Quantum extension" of classical distribution function

$$W(x,p) = \int \frac{d\,y}{2\pi\hbar} e^{-\frac{i}{\hbar}p\cdot y} \psi^* \left(x+\frac{y}{2}\right) \psi \left(x-\frac{y}{2}\right)$$

Properties: $\int dp\, W(x,p) = |\psi(x)|^2, \qquad \int dx\, W(x,p) = |\psi(p)|^2$

Connected to probability!

- Expectation value of any operator $\hat{A}$

$$\langle \hat{A} \rangle = \int dx\, dp\, W(x,p) a(x,p)$$

Wigner function - Quantum field theory

$$W_{\chi\sigma}(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle : \bar{\Psi}_\sigma \left(x+\frac{y}{2}\right) \Psi_\chi \left(x-\frac{y}{2}\right) : \right\rangle$$

- Dirac equation $\Longrightarrow$ Equation of motion for Wigner function
  Elze, Gyulassy, Vasak, Ann. Phys. 173 (1987) 462
  de Groot, van Leeuwen, van Weert, Relativistic Kinetic Theory. Principles and Applications

$$\left[\gamma\cdot\left(p+i\frac{\hbar}{2}\partial\right)-m\right] W(x,p) = \hbar \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle : \rho\left(x-\frac{y}{2}\right)\bar{\psi}\left(x+\frac{y}{2}\right) : \right\rangle$$

$\rho = -(1/\hbar)\partial\mathcal{L}_I/\partial\bar{\psi}$, $\mathcal{L}_I =$ interaction Lagrangian

$\Longrightarrow$ Boltzmann equation $\Longrightarrow$ Kinetic theory

$$p\cdot\partial W_{\chi\sigma}(x,p) = C_{\chi\sigma}$$

$\hbar$-expansion

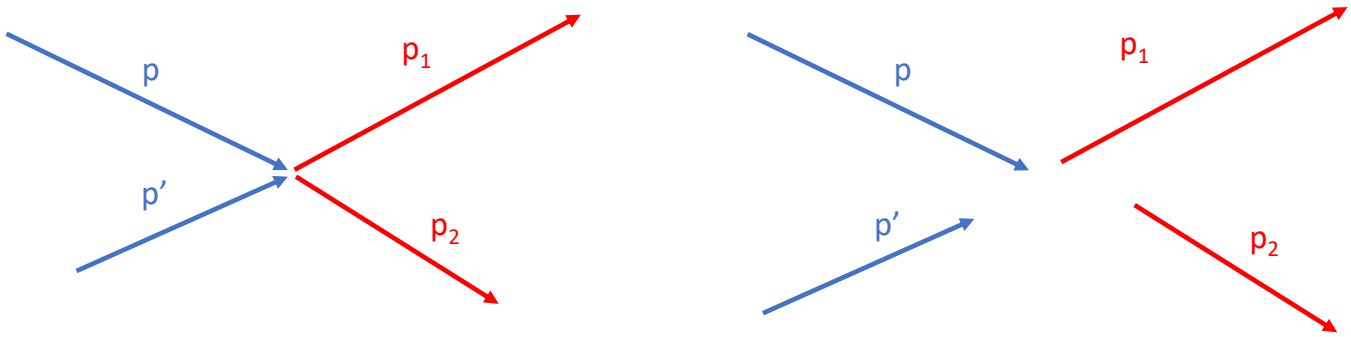
Weickgenannt, ES, Sheng, Wang, Rischke, 2005.01506 (2020), 2103.04896 (2021)

- Semiclassical expansion of Wigner function

$$W = W^{(0)} + \hbar W^{(1)} + \mathcal{O}(\hbar^2)$$

- Semiclassical and **nonlocal** expansion of collision kernel

$$C = C^{(0)}_{\text{local}} + \hbar C^{(1)}_{\text{local}} + \hbar C^{(1)}_{\text{nonlocal}} + \mathcal{O}(\hbar^2)$$



Local **Nonlocal**

Spin in phase space

Weickgenannt, ES, Sheng, Wang, Rischke, 2005.01506 (2020), 2103.04896 (2021)

- In order to account for spin dynamics enlarge phase space
  J. Zamanian, M. Marklund, and G. Brodin, NJP 12, 043019 (2010)
  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)
- Introduce new phase-space variable $\mathfrak{s}^\mu$

$$\mathfrak{f}(x,p,\mathfrak{s}) \equiv \frac{1}{2}\left[\bar{\mathcal{F}}(x,p) - \mathfrak{s}\cdot\mathcal{A}(x,p)\right]$$

- Components of Wigner fct. $\bar{\mathcal{F}} = m/p^2 \mathrm{tr}[p\cdot\gamma W]$, $\mathcal{A}^\mu = \mathrm{tr}[\gamma^\mu\gamma_5 W]$

$$\bar{\mathcal{F}} = \int dS(p)\,\mathfrak{f}(x,p,\mathfrak{s}) \qquad \mathcal{A}^\mu = \int dS(p)\,\mathfrak{s}^\mu \mathfrak{f}(x,p,\mathfrak{s})$$

with $dS(p) \equiv \frac{\sqrt{p^2}}{\sqrt{3}\pi} d^4\mathfrak{s}\,\delta(\mathfrak{s}^2+3)\delta(p\cdot\mathfrak{s})$

- Boltzmann equation

$$p\cdot\partial\,\mathfrak{f}(x,p,\mathfrak{s}) = m\,\mathfrak{C}[\mathfrak{f}]$$

All dynamics in one scalar equation!

Nonlocal collisions

Weickgenannt, ES, Sheng, Wang, Rischke, 2005.01506 (2020), 2103.04896 (2021)

$$\mathfrak{C}[\mathfrak{f}] = \mathfrak{C}_{\text{local}}[\mathfrak{f}] + \hbar\, \mathfrak{C}_{\text{nonlocal}}[\mathfrak{f}]$$

- Long calculation $\Longrightarrow$ Intuitive result in low-density approximation:

$$\mathfrak{C}[\mathfrak{f}] = \int d\Gamma_1 d\Gamma_2 d\Gamma'\, \mathcal{W}\, [\mathfrak{f}(x+\Delta_1, p_1, \mathfrak{s}_1)\mathfrak{f}(x+\Delta_2, p_2, \mathfrak{s}_2) - \mathfrak{f}(x+\Delta, p, \mathfrak{s})\mathfrak{f}(x+\Delta', p', \mathfrak{s}')]$$
$$+ \int d\Gamma_2\, dS_1(p)\, \mathfrak{W}\, \mathfrak{f}(x+\Delta_1, p, \mathfrak{s}_1)\mathfrak{f}(x+\Delta_2, p_2, \mathfrak{s}_2)$$

$d\Gamma \equiv d^4p\, dS(p)$

- Structure: Momentum and spin exchange + Spin exchange only
- Nonlocal Collisions $\Longrightarrow$ Displacement $\Delta \sim \mathcal{O}(\hbar) \sim \mathcal{O}(\partial)$
- $\mathcal{W}$, $\mathfrak{W}$ vacuum transition probabilities, depend on phase-space spins

Equilibrium distribution function

Weickgenannt, ES, Sheng, Wang, Rischke, 2005.01506 (2020), 2103.04896 (2021)

- **Equilibrium condition:** $\mathfrak{C}[\mathfrak{f}] = 0$
- Ansatz for distribution function
  F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, AP. 338, 32 (2013)
  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)

$$\mathfrak{f}_{eq}(x,p,\mathfrak{s}) \propto \exp\Big[\underbrace{-\beta(x)\cdot p}_{\text{Energy-momentum}} \underbrace{+\frac{\hbar}{4}\Omega_{\mu\nu}(x)\Sigma^{\mu\nu}_{\mathfrak{s}}}_{\text{Total angular momentum}}\Big]\delta(p^2-M^2)$$

  - M - mass (possibly modified by interactions)
  - $\beta^\mu = u^\mu/T$, Spin potential $\Omega^{\mu\nu}$
  - Spin-dipole-moment tensor $\Sigma^{\mu\nu}_{\mathfrak{s}} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha\mathfrak{s}_\beta$

- Insert into $\mathfrak{C}[\mathfrak{f}]$ and expand up to $\mathcal{O}(\hbar)$

Condition for $\mathfrak{C}[\mathfrak{f}] = 0 \Longrightarrow$ **Global equilibrium**

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0 \qquad \Omega_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu) = \text{const.}$$

System gets polarized through rotations!

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Spin hydrodynamics

Canonical energy-momentum and spin tensor

Action $\Rightarrow$ Poincaré symmetry $\Rightarrow$ Noether's th. $\Rightarrow$ Conservation laws

- **Conservation of energy and momentum:**
  Canonical energy-momentum tensor $\hat{T}_C^{\mu\nu}(x)$

$$\partial_\mu \hat{T}_C^{\mu\nu}(x) = 0$$

- **Conservation of total angular momentum:**
  Canonical total angular momentum tensor ("orbital"+"spin")

$$\hat{J}_C^{\lambda,\mu\nu}(x) = x^\mu \hat{T}_C^{\lambda\nu}(x) - x^\nu \hat{T}_C^{\lambda\mu}(x) + \hat{S}_C^{\lambda,\mu\nu}(x)$$

$$\partial_\lambda \hat{J}_C^{\lambda,\mu\nu}(x) = 0 \Longrightarrow \partial_\lambda \hat{S}_C^{\lambda,\mu\nu}(x) = \hat{T}_C^{\nu\mu}(x) - \hat{T}_C^{\mu\nu}(x)$$

Pseudo-gauge transformations

ES, Weickgenannt, 2007.00138 (Review)

Densities are not uniquely defined $\Longrightarrow$ Relocalization

F. W. Hehl, Rep. Mat. Phys. 9, 55 (1976)

$$\hat{T}'^{\,\mu\nu}(x) = \hat{T}_C^{\mu\nu}(x) + \tfrac{1}{2}\partial_\lambda \left[\hat{\Phi}^{\lambda,\,\mu\nu}(x) + \hat{\Phi}^{\mu,\,\nu\lambda}(x) + \hat{\Phi}^{\nu,\,\mu\lambda}(x)\right]$$

$$\hat{S}'^{\,\lambda,\,\mu\nu} = \hat{S}_C^{\lambda,\,\mu\nu}(x) - \hat{\Phi}^{\lambda,\,\mu\nu}(x) + \partial_\rho \hat{Z}^{\mu\nu,\lambda\rho}(x)$$

$\hat{\Phi}^{\lambda,\,\mu\nu} = -\hat{\Phi}^{\lambda,\,\nu\mu}$, $\hat{Z}^{\mu\nu,\lambda\rho} = -\hat{Z}^{\nu\mu,\lambda\rho} = -\hat{Z}^{\mu\nu,\rho\lambda}$

- Leave global charges invariant

$$\hat{P}^\mu = \int d^3\Sigma_\lambda\, \hat{T}_C^{\lambda\mu}(x) \qquad \hat{J}^{\mu\nu} = \int d^3\Sigma_\lambda\, \hat{J}_C^{\lambda,\,\mu\nu}(x)$$

- However, different global spin

$$\hat{S}_C^{\mu\nu} = \int d\Sigma_\lambda \hat{S}_C^{\lambda,\mu\nu} \neq \hat{S}'^{\mu\nu}$$

- Conservation laws $\partial_\mu \hat{T}'^{\mu\nu} = 0$, $\partial_\lambda \hat{S}'^{\lambda,\mu\nu} = \hat{T}'^{[\nu\mu]}$

- Belinfante's case ($\hat{\Phi}^{\lambda,\,\mu\nu}(x) = \hat{S}_C^{\lambda,\,\mu\nu}(x)$ and $\hat{Z}^{\lambda\rho,\mu\nu} = 0$)

$$\hat{T}_B^{\mu\nu}(x) = \hat{T}_C^{\mu\nu}(x) + \tfrac{1}{2}\partial_\lambda \left[\hat{S}_C^{\lambda,\,\mu\nu}(x) + \hat{S}_C^{\mu,\,\nu\lambda}(x) + \hat{S}_C^{\nu,\,\mu\lambda}(x)\right], \quad \hat{S}_B^{\lambda,\,\mu\nu}(x) = 0$$

Spin hydrodynamics

ES, Weickgenannt, 2007.00138 (2020) (Review)

Energy-momentum tensor: $\hat{T}^{\lambda\nu}$
Total angular momentum tensor:

$$\hat{J}^{\lambda,\mu\nu} \equiv x^\mu \hat{T}^{\lambda\nu} - x^\nu \hat{T}^{\lambda\mu} + \hbar \hat{S}^{\lambda,\mu\nu}$$

Additional dynamical tensor: Spin tensor $\hat{S}^{\lambda,\mu\nu}$

- Hydrodynamic densities from quantum field theory

$$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle \qquad S^{\lambda,\mu\nu} = \langle \hat{S}^{\lambda,\mu\nu} \rangle$$

- **10 hydro eqs.**: 4 Energy-momentum + 6 Total angular momentum cons.

$$\partial_\mu T^{\mu\nu} = 0 \qquad \hbar \partial_\lambda S^{\lambda,\mu\nu} = T^{[\nu\mu]}$$

$T^{[\nu\mu]} = T^{\nu\mu} - T^{\mu\nu}$

- **10 unknowns**: $\beta^\mu = u^\mu / T$ and $\Omega^{\mu\nu}$
- $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$ are NOT uniquely defined $\Longrightarrow$ Pseudo-gauge transformations
  - Canonical - **Problem:** Total spin is not a tensor for free fields
  - **Solution**: Hilgevoord, Wouthuysen, NP 40 (1963) 1

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Spin hydro with Hilgevoord-Wouthuysen currents

Weickgenannt, ES, Sheng, Wang, Rischke, 2005.01506 (2020), 2103.04896 (2021);
ES, Weickgenannt, 2007.00138 (2020)

- Equations of motion from kinetic theory

$$\partial_\mu T^{\mu\nu}_{\rm HW} = \int d\Gamma\, p^\nu\, \mathfrak{C}[\mathfrak{f}] = 0$$

$$\hbar\, \partial_\lambda S^{\lambda,\mu\nu}_{\rm HW} = \int d\Gamma\, \frac{\hbar}{2} \Sigma^{\mu\nu}_{\mathfrak{s}}\, \mathfrak{C}[\mathfrak{f}] = T^{[\nu\mu]}_{\rm HW}$$

$\Sigma^{\mu\nu}_{\mathfrak{s}} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha \mathfrak{s}_\beta$

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- Energy-momentum conserved in a collision

Spin-dipole $\Sigma^{\mu\nu}_{\mathfrak{s}}$ not conserved in nonlocal collisions $\Longrightarrow$ $T^{[\nu\mu]}_{\rm HW} \neq 0$
$\Longrightarrow$ Conversion between spin and orbital angular momentum

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$\Longrightarrow$ Conversion between spin and orbital angular momentum

- $T^{[\nu\mu]}_{\rm HW} = 0$: (i) for local collisions (**spin** is collisional invariant)
(ii) in global equilibrium ($\mathfrak{C}[\mathfrak{f}] = 0$)

- Nonlocal collisions away from global equilibrium $\Longrightarrow$ Dissipative dynamics

Fluid gets polarized through rotation!

Spin hydro with Hilgevoord-Wouthuysen currents

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$\Sigma^{\mu\nu}_{\mathfrak{s}} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha \mathfrak{s}_\beta$

- Energy-momentum conserved in a collision

Spin-dipole $\Sigma^{\mu\nu}_{\mathfrak{s}}$ not conserved in nonlocal collisions $\Longrightarrow$ $T^{[\nu\mu]}_{\rm HW} \neq 0$
$\Longrightarrow$ Conversion between spin and orbital angular momentum

- $T^{[\nu\mu]}_{\rm HW} = 0$: (i) for local collisions (**spin** is collisional invariant)
(ii) in global equilibrium ($\mathfrak{C}[\mathfrak{f}] = 0$)

- Nonlocal collisions away from global equilibrium $\Longrightarrow$ Dissipative dynamics

Fluid gets polarized through rotation!

- What is the meaning of local equilibrium with nonlocal collisions?

Nonrelativistic limit

- $p^\mu \to m(1, \boldsymbol{v})$, $\Sigma_{\mathfrak{s}}^{\mu\nu} \to \epsilon^{ijk}\mathfrak{s}^k$

$$T_{\mathrm{HW}}^{[ji]} = m\epsilon^{ijk}\partial^0\left\langle \frac{\hbar}{2}\mathfrak{s}^k \right\rangle + m\epsilon^{ijk}\partial^l\left\langle v^l \frac{\hbar}{2}\mathfrak{s}^k \right\rangle$$

with $\langle ... \rangle \equiv (m^2/2\pi\sqrt{3}) \int d^3v\, d^3\mathfrak{s}\, \delta(\mathfrak{s}^2 - 3)\,(...)f$

- Agreement with phenomenological result of nonrelativistic kinetic theory.
  S. Hess and L. Waldmann, Zeitschrift für Naturforschung A 26, 1057 (1971)
- Comparison with micropolar fluids - Viscous fluids made of particle with internal angular momentum with mass density ϱ and velocity u^i
  G. Lukaszewicz, Micropolar Fluids, Theory and Applications (Birkhäuser Boston, 1999)

$$\varrho\left(\partial^0 + u^j\partial^j\right)\ell^i = \partial^j C^{ji} + \epsilon^{ijk}T^{jk}$$

$\Longrightarrow$ Internal angular momentum $\varrho\,\ell^i = m\left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle$,

$\Longrightarrow$ Couple stress tensor $C^{ji} = -\left\langle \frac{\hbar}{2}\mathfrak{s}^i p^j \right\rangle + m\left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle u^j$.

- Change of internal angular momentum due to C^{ji} and $\epsilon^{ijk}T^{jk}$
- Many applications in condensed matter e.g. spintronics and chiral active fluids
  R. Takahashi, M. Matsuo, M. Ono, K. Harii, H. Chudo, S. Okayasu, J. Ieda, S. Takahashi, S. Maekawa, and E. Saitoh, Nature Physics 12, 52 (2016)
  D. Banerjee, A. Souslov, A. G. Abanov, and V. Vitelli, Nature communications 8, 1 (2017)

Polarization observable in heavy-ion collisions

- Polarization vector for particle with momentum p^μ (e.g. Λ-hyperon)

  Becattini, 2004.04050; ES, Weickgenannt, 2007.00138; Tinti, Florkowski, 2007.04029

$$\Pi_\mu(p) = \frac{\hbar}{2m} \frac{\int d\Sigma_\lambda p^\lambda \mathcal{A}_\mu}{\int d\Sigma_\lambda p^\lambda \, \mathrm{tr}[W]}$$

  $\mathcal{A}_\mu = \mathrm{tr}[\gamma_\mu \gamma_5 W]$, Σ_λ - Hypersurface

- **Equilibrium**

$$\Pi_\mu(p) = -\frac{\hbar^2}{8m} \epsilon_{\mu\nu\alpha\beta} p^\nu \frac{\int d\Sigma_\lambda p^\lambda f(1-f)\varpi^{\alpha\beta}}{\int d\Sigma_\lambda p^\lambda f}$$

  $\varpi^{\alpha\beta} = -\frac{1}{2}(\partial^\alpha \beta^\beta - \partial^\beta \beta^\alpha)$ - Thermal vorticity
  f - Distribution function

  Becattini, Chandra, Del Zanna, Grossi, Annals. Phys. 338, 32 (2013)

What are nonequilibrium effects on $\Pi_\mu(p)$?

Conclusions

- New era where spin degrees of freedom must be taken into account in hydrodynamics/kinetic theory - New experimental observables
- Vorticity and spin polarization are inherently connected
- Quantum field theory calculations suggest that spin hydrodynamics from kinetic theory with nonlocal collisions is always dissipative
- Study nonequilibrium effects on spin observables

Backup

Calculating the Wigner function

- Clifford decomposition

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right)$$

- Determine $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ from equations of motion
- Assumption: polarization effects at least $\mathcal{O}(\hbar)$

$$\mathcal{V}^\mu = \frac{1}{m}p^\mu\bar{\mathcal{F}} + \mathcal{O}(\hbar^2), \qquad \bar{\mathcal{F}} \equiv \mathcal{F} - \frac{\hbar}{m^2}p^\mu \mathrm{ReTr}(\gamma_\mu\mathcal{C})$$

- Transport equations:

$$p\cdot\partial\bar{\mathcal{F}} = m\,C_F, \qquad p\cdot\partial\mathcal{A}^\mu = m\,C_A^\mu$$

with $C_F = 2\mathrm{Im}\,\mathrm{Tr}(\mathcal{C})$, $C_A^\mu \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\nu \mathrm{Im}\,\mathrm{Tr}(\sigma_{\alpha\beta}\mathcal{C})$

Equilibrium conditions

$$\mathfrak{C}[\mathfrak{f}_{eq}] = -\int d\Gamma' d\Gamma_1 d\Gamma_2 \widetilde{\mathcal{W}} e^{-\beta\cdot(p_1+p_2)}$$

$$\times \left[\partial_\mu\beta_\nu \left(\Delta_1^\mu p_1^\nu + \Delta_2^\mu p_2^\nu - \Delta^\mu p^\nu - \Delta'^\mu p'^\nu\right) - \frac{1}{2}\Omega_{\mu\nu}\frac{\hbar}{2}\left(\Sigma_{\mathfrak{s}_1}^{\mu\nu} + \Sigma_{\mathfrak{s}_2}^{\mu\nu} - \Sigma_{\mathfrak{s}}^{\mu\nu} - \Sigma_{\mathfrak{s}'}^{\mu\nu}\right)\right]$$

$$-\int d\Gamma_2\, dS_1(p) dS'(p_2)\, \mathfrak{W}\, e^{-\beta\cdot(p+p_2)}$$

$$\times \left\{\partial_\mu\beta_\nu \left[(\Delta_1^\mu - \Delta^\mu)p^\nu + (\Delta_2^\mu - \Delta'^\mu)p_2^\nu\right] - \frac{1}{2}\Omega_{\mu\nu}\frac{\hbar}{2}(\Sigma_{\mathfrak{s}_1}^{\mu\nu} + \Sigma_{\mathfrak{s}_2}^{\mu\nu} - \Sigma_{\mathfrak{s}}^{\mu\nu} - \Sigma_{\mathfrak{s}'}^{\mu\nu})\right\}.$$

- Conservation of total angular momentum (orbital+spin) in a collision

$$j^{\mu\nu} = \Delta^\mu p^\nu - \Delta^\nu p^\mu + \frac{\hbar}{2}\Sigma_{\mathfrak{s}}^{\mu\nu}$$

- Conditions for vanishing collision term

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0$$

$$\Omega_{\mu\nu} = \varpi_{\mu\nu} \equiv -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu) = \text{const.}$$

Global equilibrium!

Canonical - Free Dirac theory

$$\hat{T}^{\mu\nu}_C(x) = \frac{i\hbar}{2}\bar{\psi}(x)\gamma^\mu \overleftrightarrow{\partial}^\nu \psi(x) - g^{\mu\nu}\mathcal{L}(x)$$

$$\hat{S}^{\lambda,\mu\nu}_C(x) = \frac{\hbar}{4}\bar{\psi}(x)(\gamma^\lambda\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\lambda)\psi(x) = -\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}\gamma_\alpha\gamma_5\psi$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

- Global spin

$$\hat{S}^{\mu\nu}_C \equiv \int_\Sigma d\Sigma_\lambda\, \hat{S}^{\lambda,\mu\nu}_C$$

  does not transform as a tensor since $\partial_\lambda \hat{S}^{\lambda,\mu\nu}_C \neq 0$
- In a general frame: $\hat{S}^{0i} = 0$ and $\hat{S}^{ij} = \epsilon^{ijk}\hat{S}^k_C$

$$\hat{S}^k_C = \int d^3x\, \psi^\dagger \frac{\hbar}{2}\mathfrak{S}^k\psi, \quad \mathfrak{S}^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

We require covariant description of spin for free fields

Hilgevoord-Wouthuysen currents

J. Hilgevoord, S. Wouthuysen, NP 40 (1963) 1

- Apply Noether's theorem to Klein-Gordon Lagrangian for free spinors

$$\mathcal{L}_{KG} = \frac{1}{2m}(\hbar^2 \partial_\mu \bar{\psi} \partial^\mu \psi - m^2 \bar{\psi}\psi)$$

and use Dirac equation as subsidiary condition

$$\hat{T}^{\mu\nu}_{HW} = \frac{\hbar^2}{2m}\left(\partial^\mu \bar{\psi}\partial^\nu \psi + \partial^\nu \bar{\psi}\partial^\mu \psi\right) - g^{\mu\nu}\mathcal{L}_{KG}$$

$$\hat{S}^{\lambda,\mu\nu}_{HW} = \frac{i\hbar^2}{4m}\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^\lambda \psi$$

- $\partial_\lambda \hat{S}^{\lambda,\mu\nu}_{HW} = 0 \Longrightarrow$ Covariant global spin $\hat{S}^{\mu\nu}_{HW} \equiv \int_\Sigma d\Sigma_\lambda \, \hat{S}^{\lambda,\mu\nu}_{HW}$
- Compatible with Frenkel theory $\hat{P}_\mu \hat{S}^{\mu\nu}_{HW} = 0$
- Pseudo-gauge transformation with

$$\hat{\Phi}^{\lambda,\mu\nu} = \hat{M}^{[\mu\nu]\lambda} - g^{\lambda[\mu}\hat{M}_\rho{}^{\nu]\rho}$$

$$\hat{Z}^{\mu\nu,\lambda\rho} = -\frac{\hbar}{8m}\bar{\psi}(\sigma^{\mu\nu}\sigma^{\lambda\rho} + \sigma^{\lambda\rho}\sigma^{\mu\nu})\psi$$

where $\hat{M}^{\lambda\mu\nu} \equiv \frac{i\hbar^2}{4m}\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^\lambda \psi$

Statistical operator - Canonical

Zubarev, 1979, Ch, Van Weert 1982 , F. Becattini, L. Bucciantini, E. Grossi and L. Tinti, Eur. Phys. J. C **75**, no. 5, 191 (2015), F. Becattini, M. Buzzegoli, E. Grossi, Particles 2 (2019) 2

- Maximization of entropy

$$S = -\mathrm{tr}(\hat{\rho}_C \log \hat{\rho}_C)$$

- Constraints on energy and momentum

$$n_\mu \mathrm{tr}\left[\hat{\rho}_C\, \hat{T}^{\mu\nu}_C(x)\right] = n_\mu T^{\mu\nu}_C(x)$$

n^μ - vector orthogonal to hypersurface Σ

Spin tensor $\Longrightarrow$ Constraint on total angular momentum

$$n_\mu \mathrm{tr}\left(\hat{\rho}_C\, \hat{J}^{\mu,\lambda\nu}_C\right) = n_\mu \mathrm{tr}\left[\hat{\rho}_C \left(x^\lambda \hat{T}^{\mu\nu}_C - x^\nu \hat{T}^{\mu\lambda}_C + S^{\mu,\lambda\nu}_C\right)\right] = n_\mu J^{\mu,\lambda\nu}_C$$

- Density operator

$$\begin{aligned}\hat{\rho}_C =& \frac{1}{Z} \exp\left[-\int_\Sigma d\Sigma_\mu \left(\hat{T}^{\mu\nu}_C(x) b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\lambda\nu}_C(x)\Omega_{\lambda\nu}(x)\right)\right]\\ =& \frac{1}{Z} \exp\left[-\int_\Sigma d\Sigma_\mu \left(\hat{T}_C(x)^{\mu\nu} \beta_\nu(x) - \frac{1}{2}\hat{S}^{\mu,\lambda\nu}_C(x)\Omega_{\lambda\nu}(x)\right)\right]\end{aligned}$$

$\Omega_{\lambda\nu}$ - Spin potential: Lagrange multiplier for conservation of total angular momentum

Global equilibrium - Canonical

- Asymmetric EM tensor $\hat{T}_C^{\mu\nu} = \hat{T}_S^{\mu\nu} + \hat{T}_A^{\mu\nu}$ with $\hat{T}_S^{\mu\nu} = \hat{T}_S^{\nu\mu}$, $\hat{T}_A^{\mu\nu} = -\hat{T}_A^{\nu\mu}$
- Density operator must be stationary

$$\frac{1}{2}\hat{T}_S^{\mu\nu}(\partial_\mu\beta_\nu+\partial_\nu\beta_\mu)+\frac{1}{2}\hat{T}_A^{\mu\nu}(\partial_\mu\beta_\nu-\partial_\nu\beta_\mu)-\frac{1}{2}(\partial_\mu\hat{S}_C^{\mu,\lambda\nu})\Omega_{\lambda\nu}-\frac{1}{2}\hat{S}_C^{\mu,\lambda\nu}(\partial_\mu\Omega_{\lambda\nu}) = 0$$

- Global equilibrium conditions:

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0$$
$$\beta_\nu = b_\nu + \Omega_{\nu\lambda}x^\lambda$$
$$\Omega_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu) = \text{const}$$

We used $\partial_\mu\hat{S}_C^{\mu,\lambda\nu} = -2\hat{T}_A^{\lambda\nu}$

F. Becattini, Phys. Rev. Lett. **108**, 244502 (2012)

Statistical operator and pseudo-gauges

F. Becattini, W. Florkowski, E.S. PLB **789**, 419 (2019)

- Start with **Canonical**

$$\hat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\hat{T}_C^{\mu\nu}\beta_\nu - \frac{1}{2}\hat{S}_C^{\mu,\lambda\nu}\Omega_{\lambda\nu}\right)\right]$$

- Use general transformation $(\hat{T}_C^{\mu\nu}, \hat{S}_C^{\lambda,\mu\nu}) \to (\hat{T}'^{\mu\nu}, \hat{S}'^{\lambda,\mu\nu})$:
- **Canonical** density operator becomes

$$\hat{\rho}_C = \frac{1}{Z}\exp\Bigg[-\int_\Sigma d\Sigma_\mu \Big(\hat{T}'^{\mu\nu}\beta_\nu - \hat{S}'^{\mu,\lambda\nu}\Omega_{\lambda\nu} + \frac{1}{2}\xi_{\lambda\nu}\left(\hat{\Phi}^{\lambda,\mu\nu} + \hat{\Phi}^{\nu,\mu\lambda}\right) - \frac{1}{2}(\Omega_{\lambda\nu} - \varpi_{\lambda\nu})\hat{\Phi}^{\mu,\lambda\nu} - \hat{Z}^{\lambda\nu,\mu\rho}\partial_\rho\Omega_{\lambda\nu}\Big)\Bigg]$$

with $\varpi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda - \partial_\lambda\beta_\nu)$, $\xi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda + \partial_\lambda\beta_\nu)$

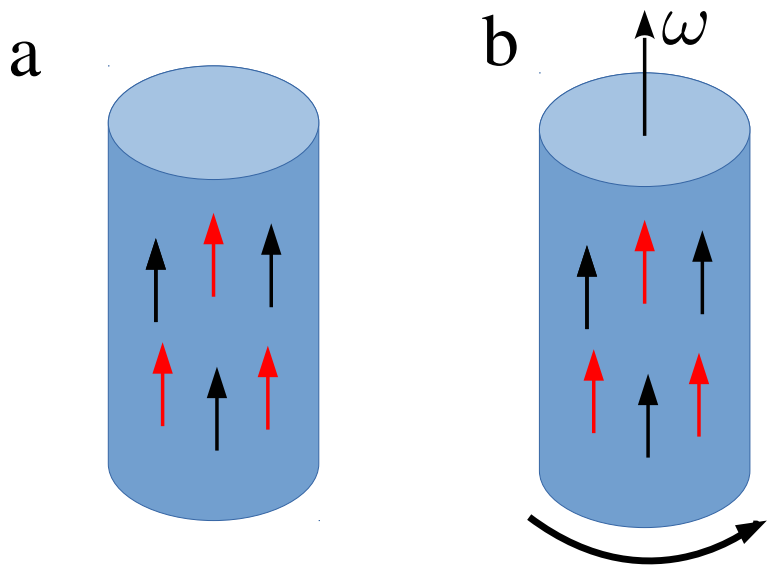
- When is $\hat{\rho}_C = \hat{\rho}'$?

$$\hat{\rho}' = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\hat{T}'^{\mu\nu}\beta_\nu - \frac{1}{2}\hat{S}'^{\mu,\lambda\nu}\Omega_{\lambda\nu}\right)\right]$$

1. β_μ is the same in both cases
2. $\xi_{\lambda\nu} = 0$ or $\hat{S}_C^{\lambda,\mu\nu} + \hat{S}_C^{\nu,\mu\lambda} = 0$
3. $\Omega_{\lambda\nu}$ coincides with thermal vorticity, $\Omega_{\lambda\nu} = \varpi_{\lambda\nu} =$ constant

Equivalence in global equilibrium!

Polarized neutral system - Physical meaning of $\Omega_{\mu\nu}$



- a) Fluid at rest with constant temperature with particles and antiparticles polarized in the same direction $\beta^\mu = (1/T)(1, \mathbf{0}) \Rightarrow \varpi = \frac{1}{2}(\partial_\nu \beta_\lambda - \partial_\lambda \beta_\nu) = 0$; b) Polarized system with rotation
- Belinfante's pseudo-gauge does not imply that polarization vanishes, but rather it is locked to thermal vorticity
- Spin tensor and spin potential to describe hydro evolution, but only if spin density relaxes "slowly" compared to the microscopic interaction scale
- In general, away from equilibrium $\Omega_{\lambda\nu} \neq \varpi_{\lambda\nu}$
- It is crucial to calculate spin relaxation times see e.g. J. Kapusta, E. Rrapaj, S. Rudaz PRC 101, 024907; S. Li, H. U. Yee PRD 100, 056022; D.L. Yang, K. Hattori, Y. Hidaka, JHEP 07 (2020) 070