University of Alabama Department of Physics & Astronomy
Graduate Qualifying Exam
Part 1: Classical Mechanics

January 2021

General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 150 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. Three particles, A, B and C, each having mass m, form a **rigid** structure. At time $t = 0$ they are located at $(x, y, z) = (0, 0, 0), (a, 0, 0)$ and $(0, a, 0)$, respectively, where (x, y, z) denote Cartesian coordinates.

   (a) (**4 points**) What is the location of the center of mass at time $t = 0$?

   (b) (**5 points**) What is the moment of inertia about the center of mass along the $z-$axis?

   Assume the system is at rest at time $t = 0$. Say particle A experiences a constant external force of magnitude f which is always directed towards B

   (c) (**5 points**) By what angle θ has particle A rotated about the center of mass at time t?

   (d) (**6 points**) Find an integral expression for the velocity of particle A at time t.

2. A point mass is constrained to move on a hoop of radius R which is rotating with a constant angular velocity ω about the vertical axis, as shown below.

   (a) (**14 points**) Obtain the Euler-Lagrange equation of motion describing the behavior of the angle θ, assuming that the only external force is due to gravity.

   (b) (**6 points**) Show that if ω is greater than a critical value, the particle can be at an equilibrium point on the hoop other than at the bottom (and the top, which of course is unstable). What is the critical value?

3. (**20 points**) In a coordinate system rotating with angular velocity vector $\vec{\Omega}$ relative to an inertial frame, the motion of a particle (mass m, position vector $\vec{r}$) appears to be affected by centrifugal and Coriolis forces, as given by:

$$\vec{F}_{\rm CF} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}, \quad \vec{F}_{\rm Cor} = 2m\dot{\vec{r}} \times \vec{\Omega}$$

Consider a particle released from rest at a height h above the Earth's surface, at a latitude λ. Assume a constant vertical acceleration due to gravity g. By the time the particle reaches the Earth's surface, find the horizontal deflection of the particle away the straight line directly toward the Earth's center, due to the rotation of the Earth. Write your answer in terms of h, λ and the rotational angular velocity of the Earth Ω (i.e. $2\,\pi$ radians/day)

4. (**20 points**) In two-dimensional space, consider the motion of a particle with mass (m) under a central potential $V(r) = -\frac{1}{2}\,k\,r^2$ with a constant $k > 0$ in polar coordinates, $\{r, \theta\}$. The motion of the particle is restricted on a trajectory described as $r = r_0 e^{\frac{\theta}{\pi}}$ with a constant $r_0 > 0$. Find the position of the particle at time $t > 0$ for the initial conditions, $r = r_0$, $\frac{dr}{dt} = 0$, $\theta = 0$ and $\frac{d\theta}{dt} = 0$, at $t = 0$.

5. Consider a one-dimensional system defined by the following Lagrangian:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2,$$

where $x(t)$ is the position of a particle with mass m, and $\omega_0 > 0$ is a constant. We generalize the system by introducing a time-dependence for the mass parameter as

$$m(t) = m_0\, e^{2\frac{t}{\tau}},$$

where $m_0 > 0$ and $\tau > 1/\omega_0$ are constants.

(1) (**4 points**) In the limit of $\tau \to \infty$, find $x(t)$ that satisfies the initial conditions, $x = x_I$ and $\dot{x} = 0$, at $t = 0$.

(2) (**8 points**) For a finite value of τ, derive the Euler-Lagrange equation and find the solution satisfying the initial conditions, $x = x_I$ and $\dot{x} = 0$, at $t = 0$.

(3) (**8 points**) Using a new variable $y(t) = e^{\frac{t}{\tau}}x(t)$, show that the system is equivalent to the one-dimensional harmonic oscillator with a frequency, $\omega = \sqrt{\omega_0^2 - \frac{1}{\tau^2}}$.

6. A projectile of mass m is thrown vertically upward with initial speed v_0, in a medium where it experiences a velocity-dependent drag force with magnitude bv, where v is the speed and b is the drag coefficient.

   (a) (**12 points**) Find the expression for the velocity of this particle as a function of time

   (b) (**4 points**) Find the position of this particle as a function of time

   (c) (**2 points**) Find the time taken by the particle to reach its maximum height

   (d) (**2 points**) Show that the maximum height reached by the particle is given by:

$$y_{max} = \frac{mv_0}{b} - \frac{m^2 g}{b^2}\ln\left(1 + \frac{bv_0}{mg}\right)$$

University of Alabama Department of Physics & Astronomy
Graduate Qualifying Exam
Part 2: Electricity & Magnetism

January 2021

General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 150 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
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1. Two charges q and $-q$ are located at $(x, y, z) = (0, 0, a)$ and $(a, 0, a)$, respectively, and a perfect conductor is located in the region $z \leq 0$.

   a) (**7 points**) Find the resulting scalar potential $\Phi(x, y, z)$ in the region $z > 0$.

   b) (**8 points**) Find the resulting electric field $\vec{E}(x, y, z)$ in the region $z > 0$.

   c) (**5 points**) Find the charge density $\sigma(x, y)$ induced on the surface of the conductor.

2. (**20 points**) An infinite straight wire lies along the $x-$axis and carries a current I in the $+x$ direction. A square loop of wire, with sides of length a lies in the xy plane in the region $x, y > 0$, with sides parallel to the $x-$ and $y-$axes. If the loop moves in the $+y$ direction with a constant speed v, what is the emf induced in the square loop, and in which direction does current flow around the loop?

3. (**20 points**) The plates of a flat, parallel-plate capacitor have surface A and separation $h \ll \sqrt{A}$. Find the force between the plates, both for an isolated capacitor (as a function of the constant charge Q), and for a capacitor connected to an ideal voltage source (as a function of the constant voltage V). In both cases, use two different methods, i.e., calculate the force: (a) from the electrostatic pressure on the surface of the plates and (b) from the expression of the energy as a function of the distance between the plates.

4. (**20 points**) Consider two long concentric metal cylinders of radius a and b, separated by material which has conductivity σ. If the two cylinders are maintained at a potential difference V, find the current per unit length that flows between the cylinders (in terms of V, σ, a and b).

5. In the vacuum, the scalar and vector (retarded) potentials ($\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$) created by a charge density $\rho(\mathbf{x}, t)$ and a current $\mathbf{i}(\mathbf{x}, t)$ are given by (in the Lorentz gauge)

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3y \, \frac{\rho(\mathbf{y}, s)}{|\mathbf{x} - \mathbf{y}|}, \quad \mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3y \, \frac{\mathbf{i}(\mathbf{y}, s)}{|\mathbf{x} - \mathbf{y}|},$$

where $s = t - \frac{|\mathbf{x}-\mathbf{y}|}{c}$ with the speed of light c. Consider the case that ρ and $\mathbf{i}$ are created by a particle with electric charge q, moving in x-direction with a constant speed v_0:

$$\rho(\mathbf{x}, t) = q\, \delta^3(\mathbf{x} - \mathbf{z}(t)), \quad \mathbf{i}(\mathbf{x}, t) = q\, \mathbf{v}\, \delta^3(\mathbf{x} - \mathbf{z}(t)),$$

where $\mathbf{z}(t) = (v_0\, t, 0, 0)$ is the position of the point charge, and $\mathbf{v} = \frac{d\mathbf{z}}{dt} = (v_0, 0, 0)$.

(1) (**8 points**) Give the explicit forms for $\phi(x, 0, 0, t)$ and $\mathbf{A}(x, 0, 0, t)$ for $x > v_0\, t$.

(2) (**5 points**) Consider an inertial frame K' in which the point charge is at rest. Express the scalar and vector potentials in terms of the coordinates $(\mathbf{x}', t')$ in the inertial frame K'.

(3) (**7 points**) The Lorentz transformation between the original inertial frame and the inertial frame K' is given by

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \frac{1}{\sqrt{1 - (v_0/c)^2}} \begin{bmatrix} 1 & -v_0/c \\ -v_0/c & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix},$$

$y' = y$ and $z' = z$. Using the fact that the scalar and vector potentials in the K' frame, $\{c\phi'(\mathbf{x}', t'), \mathbf{A}'(\mathbf{x}', t')\}$ are related to those in the original frame $\{c\phi_(\mathbf{x}, t), \mathbf{A}(\mathbf{x}, t)\}$ by the same Lorentz transformation between $\{ct', \mathbf{x}'\}$ and $\{ct, \mathbf{x}\}$, find the explicit form of $\phi(\mathbf{x}, t)$ from the solution of (2).

6. A system of electric and magnetic fields can be described by a scalar potential ϕ and a vector potential $\vec{A}$

   (a) (**7 points**) Given the following set of time-dependent potentials, where q is the charge, find the associated electric and magnetic fields:

   $$\phi(\vec{r}, t) = 0 \quad \text{and} \quad \vec{A}(\vec{r}, t) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{\mathbf{r}}$$

   (b) (**6 points**) However, the set of potentials that describe a particular system is not unique. For any ϕ and $\vec{A}$, show that an alternative set of potentials ϕ' and $\vec{A}'$, also give rise to the same physical electric and magnetic fields, provided that they are related to the original potential by a gauge transformation:

   $$\phi' = \phi - \frac{\partial \lambda}{\partial t}$$

   $$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

   where $\lambda(\vec{r}, t)$ can be any scalar field.

   (c) (**6 points**) Find a gauge transformation that will allow you to write the vector potential for the system from part (a) as $\vec{A}' = 0$, and find the scalar potential in this gauge.

   (d) (**1 point**) What physical system is described by these potentials?

University of Alabama Department of Physics & Astronomy
Graduate Qualifying Exam
Part 3: Quantum Mechanics

January 2021

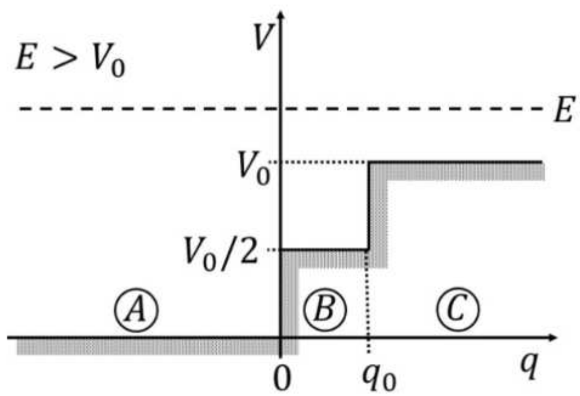
General Instructions

- Do any 5 of the 6 questions. Indicate clearly which 5 questions that you wish to have graded. Each question is worth 20 points.
- 150 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. Consider a one dimensional particle wave with energy E. The incoming wave function is given by $\varphi_0(q) = \exp(ik_0 q)$ with $k_0^2 = \frac{2m}{\hbar^2}E$ and mass m. It approaches from left towards a double potential barrier, which is given by the following function

$$V(q) = \begin{cases} 0 \text{ if } q \leq 0, & \text{zone } A \\ \frac{V_0}{2} \text{ if } 0 < q < q_0, & \text{zone } B \\ V_0 \text{ if } q_0 \leq q & \text{zone } C. \end{cases}$$

Here q is the position, and V_0 is a positive real valued constant. The energy of the incoming particle wave fulfills the following inequality $E > V_0$, as also indicated in the figure.



(a) (**1 point**) What can you conclude for the resulting particle wave function φ from the fact that $E > V_0$?

(b) (**19 points**) Determine the reflection coefficient for this problem.

2. Consider an electron of mass m with the following wave function

$$|\Psi(\vec{r},t)|^2 = \frac{1}{(\pi b^2(t))^{\frac{3}{2}}} \exp\left(-\frac{(\vec{r}-\vec{v}_0 t)^2}{b^2(t)}\right),$$

with the time dependent function

$$b(t) = b\sqrt{1+\frac{\hbar^2 t^2}{m^2 b^4}}.$$

Here t is the time, $\vec{r}$ is the position vector, $\vec{v}_0$ is the initial velocity vector, and b is a real valued constant.

(a) (**10 points**) Show that the total probability of finding the electron somewhere in space is always one at any given point in time.

(b) (**10 points**) The vector $\vec{r}^*(t)$ indicates the most probable place to find the electron in space at any given time. Determine $\vec{r}^*(t)$ for the electron.

The following mathematical equality might be helpful:

$$\int_{-\infty}^{+\infty} dx \exp\left(-x^2\right) = \sqrt{\pi}.$$

3. The wavefunction for a particle in a one-dimensional infinite square well with boundaries at $\mathbf{x} = -\mathbf{a}$ and x = a is

$$\psi(x) = \frac{\sqrt{15}}{4a^{5/2}}(a^2 - x^2)\ ,$$

in the region $-a \leq x \leq a$.

(a) (**8 points**) Compute the uncertainties in position and momentum measurements and verify that the Heisenberg uncertainty principle is satisfied.

(b) (**6 points**) What is the expectation value of the energy and how does that compare with the value of the ground state energy?

(c) (**6 points**) What is the probability that the particle is in the ground state of the well?

Integral table: $\int_{-1}^{1} dy \cos^2 \frac{n\pi y}{2} = \int_{-1}^{1} dy \sin^2 \frac{n\pi y}{2} = 1\ , \qquad n = 1, 2, 3, ...$

$\int_{-1}^{1} dy\,(1-y^2)\cos\frac{\pi y}{2} = \frac{32}{\pi^3} \qquad \int_{-1}^{1} dy\,(1-y^2)\sin\frac{n\pi y}{2} = 0$

4. Consider a quantum mechanical system characterized by two energy eigenstates,

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

corresponding to the eigenvalues E_1 and E_2 ($0 < E_1 < E_2$), respectively. Suppose that we can observe the system only through two "flavor" eigenstates A and B given by

$$|\psi_A\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad \text{and} \quad |\psi_B\rangle = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}.$$

When the flavor of the system is measured as A at time $t = 0$, answer the following questions:

(a) (**10 points**)Find the wave function $|\psi(t)\rangle$ at $t > 0$.

(b) (**10 points**)Calculate the probability to find the flavor state B at $t > 0$.

5. Consider a particle mass m in a one-dimensional quantum harmonic oscillator of frequency ω, such that the potential is $\frac{1}{2}m\omega^2x^2$.

(a) **(1 point)** What is the energy of the ground state?

(b) **(7 points)** Construct the Schrodinger Equation for this system and solve it to find the wave function for the ground state.

(c) **(12 points)** Now consider a perturbation due to a potential λx^4, where this term can be considered small compared to the oscillator potential. Find the first-order shift in energy of the ground state due to this perturbation.

Note: you may find the following integral helpful:

$$\int_{-\infty}^{+\infty} dx \exp\left(-\alpha x^2\right) = \sqrt{\frac{\pi}{\alpha}}$$

6. The matrix

$$S_{\hat{\mathbf{n}}} = \frac{1}{\sqrt{2}}(S_y + S_z) = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$$

represents the spin of an electron in the y-z plane, specifically in the $\hat{\mathbf{n}} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ direction. (The spin-$\frac{1}{2}$ matrices S_x, S_y and S_z are given below.)

(a) (**3 points**) Show that the eigenvalues of this matrix are $\pm\frac{\hbar}{2}$.

(b) (**6 points**) If the spin in the $\hat{\mathbf{n}}$ direction is measured to be $\frac{\hbar}{2}$ at time $t = 0$, what is the state of the system immediately after the measurement.

(c) (**5 points**) If the electron is in a magnetic field B in the $z-$direction, what is the state of the system at a time $t > 0$? (Assume no additional measurement is made, and that the Hamiltonian is $H = -\gamma \mathrm{B} S_z$.)

(d) (**6 points**) What is the probability that the spin in the $\hat{\mathbf{n}}$ direction is $\frac{\hbar}{2}$ at time $t > 0$?

Note: the Spin-$\frac{1}{2}$ matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

University of Alabama Department of Physics & Astronomy
Graduate Qualifying Exam
Part 4: Thermal Physics

January 2021

General Instructions

- Do any 2 of the 3 questions. Indicate clearly which 2 questions that you wish to have graded. Each question is worth 20 points.
- 90 minutes are allocated for this exam.
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your assigned number and the subject - do **not** write your name.
- Turn in this question sheet with your answer booklet.
- **No electronic devices are allowed. In particular, cell phones, smartphones and other handheld devices are explicitly prohibited.**

1. In the Big-Bang Theory, the early Universe consisted of thermal plasmas of elementary particles, and the Universe expands adiabatically. If relativistic particles, such as photons (γ), electrons (e^-), positrons (e^+), and three types of neutrino (ν_e, ν_μ and ν_τ), are in thermal equilibrium, their entropy densities are given by

$$s_\gamma(T_\gamma) = \frac{4\pi^2}{45}\, T_\gamma^3, \quad s_f(T_f) = \frac{4\pi^2}{45}\left(\frac{7}{8}\right) T_f^3,$$

where $f = e^-, e^+, \nu_{e,\mu,\tau}$, and T_a is an individual temperature of the corresponding particle "a". The entropy density of a non-relativistic particle in thermal plasma is negligibly small compared to that of a relativistic particle.

(a) (**5 points**) As the Universe expands, how does the temperature of the thermal plasma depend on the volume?

For a temperature T of the Universe between $T_D \simeq 10^{10}$ K and 10^{12} K, γ, $e^\pm$ and $\nu_{e,\mu,\tau}$ were in thermal equilibrium and they were only the relativistic particles in the thermal plasma. Due to their very weak interactions with other particles, $\nu_{e,\mu,\tau}$ decoupled from the thermal plasma of the $\gamma - e^\pm$ system at T_D. Although $e^\pm$ remained in thermal equilibrium with photons for $T_\gamma = T_{e^\pm} < T_D$, they became non-relativistic at $T_\gamma \simeq 10^9$ K. Consider that the entropies of the $\gamma - e^\pm$ system and the decoupled neutrino systems are individually conserved during the expansion of the Universe.

(b) (**5 points**) Write down an expression for the total entropy of the $\gamma - e^\pm$ plasma when $T = T_D$ and when $T_\gamma < 10^9$ K

(c) (**5 points**) Write down an expression for the total entropy of the ν_e plasma when $T = T_D$ and when $T_\nu < T_D$

(d) (**5 points**) Show that the temperatures T_{ν_e} and T_γ have the following relation for $T_\gamma < 10^9$ K:

$$\frac{T_{\nu_e}}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}.$$

2. The entropy for a certain thermodynamical system with N particles, energy E, and volume V is given by

$$S = CNV^2 \log(E/E_0) \ ,$$

where C is a constant and E_0 is a reference energy scale. Find expressions as a function of N, E and V for

a) (**4 points**) the temperature of the system

b) (**4 points**) the pressure of the system

c) (**4 points**) the chemical potential of the system

d) (**4 points**) the heat capacity at constant volume (and constant N)

e) (**4 points**) $\left.\frac{\partial P}{\partial V}\right|_{T,N}$

3. Consider an engine constructed using N particles of a monatomic ideal gas as the working substance, and progressing through the following steps:
$A \to B$: isothermal expansion at temperature T_H from a smaller volume V_S to a larger volume V_L;
$B \to C$: a decrease in temperature from T_H to T_C at constant volume V_L;
$C \to D$: isothermal compression at temperature T_C from V_L back to initial volume V_S;
$D \to A$: an increase in temperature from T_C back to original T_H at constant volume V_S;

(a) (**4 points**) Sketch this cycle on a $P - V$ diagram

(b) (**6 points**) Find the net work done in a complete cycle by this engine, in terms of T_H, T_C, V_s and V_L

(c) (**6 points**) Find an expression for the efficiency of this engine, in terms of T_H, T_C, V_s and V_L

Now suppose that a particular engine of this type is setup such that the smaller volume is a specific fraction of the larger volume, $V_S = V_L/e$, and that the colder temperature is a specific fraction of the hotter temperature, $T_C = T_H/3$.

(d) (**1 point**) For this specific engine, evaluate numerically the efficiency

(e) (**3 points**) Now evaluate numerically the efficiency of a Carnot engine operating between the same temperature range, compare the numerical efficiencies of the two engines, and comment briefly on your result.