T-duality and Double field theory

1. T-duality in the string theory

Consider a closed bosonic string propagating in the background spacetime $\mathbb{R}^{1,d} \times T^{\tilde{d}}$, where $\mathbb{R}^{1,d}$ is flat, and $\tilde{d}$-dimensional torus $T^{\tilde{d}}$ is compacted. The string coordinates can be described by

$$X^I = (X^\mu, X^a), \quad \mu = 0, 1, \cdots, d \quad \text{and} \quad a = d+1, \cdots, d+\tilde{d},$$

where the indices μ and a indicate the flat and compact dimensions respectively.

For example, for the case of the background geometry $\mathbb{R}^{1,24} \times S^1$ with only one dimension being compacted, the periodic boundary condition for this compact dimension is modified by

$$X^{25}(\tau, \sigma+\pi) = X^{25}(\tau, \sigma) + 2\pi R w^{25}, \quad w^{25} \in \mathbb{Z}$$

where w^{25} is the winding number which characterizes the number of times of closed string wrapping around the compact 25-dimension.

The mode expansion of $X^{25}(\tau,\sigma)$ is given by

$$X^{25}(\tau,\sigma) = x^{25} + l_s^2 p^{25}\tau + 2Rw^{25}\sigma + \text{oscillation modes}$$

where l_s is the string length and identify $l_s^2 = 2\alpha'$.

Explicitly, express $X^{25}(\tau,\sigma) = X_L^{25}(\tau+\sigma) + X_R^{25}(\tau-\sigma)$ with

$$X_L^{25}(\tau+\sigma) = \frac{1}{2}(x^{25}+c) + (\alpha' p^{25} + Rw^{25})(\tau+\sigma) + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-i2n(\tau+\sigma)}$$

and

$$X_R^{25}(\tau-\sigma) = \frac{1}{2}(x^{25}-c) + (\alpha' p^{25} - Rw^{25})(\tau-\sigma) + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n}\bar{\alpha}_n^\mu e^{-i2n(\tau-\sigma)}$$

where c is a constant.

Since the phase factor $\exp\{ip^{25}X^{25}\}$ should be single-valued along the compact circle S^1 of radius R, then the momentum p^{25} is quantized

$$p^{25} = \frac{K^{25}}{R}, \quad K^{25} \in \mathbb{Z}$$

where K^{25} is the Kaluza-Klein excitation number (or KK momentum by setting $R=1$).

Further, one can identify the zero modes with

$$\begin{cases} \sqrt{2\alpha'}\,\alpha_0^{25} = \alpha' p^{25} + R w^{25} & \text{for the left-mover} \\ \sqrt{2\alpha'}\,\bar{\alpha}_0^{25} = \alpha' p^{25} - R w^{25} & \text{for the right-mover} \end{cases}$$

The Virasoro generators are given by

$$\begin{cases} L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha^I_{m-n}\alpha_{I,n}: \\ \bar{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\bar{\alpha}^I_{m-n}\bar{\alpha}_{I,n}: \end{cases}$$

$$\therefore\ L_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha^I_{-n}\alpha_{I,n}: = \frac{1}{2}(\alpha_0^I)^2 + \sum_{n=1}^{\infty} :\alpha^I_{-n}\alpha_{I,n}:$$

$$= \frac{1}{2}\left[(\alpha_0^\mu)^2 + (\alpha_0^{25})^2\right] + N_L, \quad \mu = 0, 1, \cdots, 24$$

where $N_L = \sum_{n=1}^{\infty} :\alpha^I_{-n}\alpha_{I,n}$ is the number operator for the left-mover.

Similarly, one gets

$$\bar{L}_0 = \frac{1}{2}\left[(\bar{\alpha}_0^\mu)^2 + (\bar{\alpha}_0^{25})^2\right] + N_R$$

with $N_R = \sum_{n=1}^{\infty} :\bar{\alpha}^I_{-n}\bar{\alpha}_{I,n}$ for the right-mover.

By setting $\alpha_0^\mu = \bar{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$, for the 25-dimensional physics, the 25-dimensional mass squared is

$$m^2 = -p^\mu p_\mu = -\frac{4}{\alpha'^2}(\alpha_0^\mu)^2 = -\frac{4}{\alpha'^2}(\bar{\alpha}_0^\mu)^2, \quad \mu = 0, 1, \cdots, 24$$

Due to the vanishing of the worldsheet energy-momentum tensor, the on-shell physical string state should satisfy the level-matching condition $(L_0 - 1)|\psi\rangle = (\bar{L}_0 - 1)|\psi\rangle = 0$

Therefore,

$$-\frac{\alpha'}{4}m^2 + \frac{1}{2}(\alpha_0^{25})^2 + N_L - 1 = -\frac{\alpha'}{4}m^2 + \frac{1}{2}(\bar{\alpha}_0^{25})^2 + N_R - 1$$

$$\Rightarrow N_R - N_L = \frac{1}{2}\left[(\alpha_0^{25})^2 - (\bar{\alpha}_0^{25})^2\right] = \frac{1}{2}(\alpha_0^{25} + \bar{\alpha}_0^{25})(\alpha_0^{25} - \bar{\alpha}_0^{25})$$

By using $\begin{cases} \alpha_0^{25} = \frac{1}{\sqrt{2\alpha'}}\left(\alpha'\frac{K^{25}}{R} + RW^{25}\right) \\ \bar{\alpha}_0^{25} = \frac{1}{\sqrt{2\alpha'}}\left(\alpha'\frac{K^{25}}{R} - RW^{25}\right) \end{cases}$

one gets

$$N_R - N_L = \frac{1}{2}\left(\sqrt{2\alpha'}\frac{K^{25}}{R}\right)\left(\sqrt{\frac{2}{\alpha'}}RW^{25}\right) = K^{25}W^{25}$$

So, the level-matching condition can be rewritten as

$$L_0 - \bar{L}_0 = N_R - N_L - K^{25}W^{25} = 0$$

The 25-dimensional mass spectrum is given by

$$m^2 = \frac{2}{\alpha'}(N_L + N_R - 2) + \left[\left(\frac{K^{25}}{R}\right)^2 + \left(\frac{w^{25}}{\alpha'}R\right)^2\right]$$

The T-duality is a symmetry which makes the above mass spectrum invariant under the exchanges

$$w^{25} \to K^{25} \quad \text{and} \quad R \to \frac{\alpha'}{R}$$

2. Double field theory

The string field is defined as a state vector in the Hilbert space $\mathcal{H}$ of the corresponding 2-D CFT. Its component with respect to the basis of $\mathcal{H}$ is called the target space field.

For the bosonic closed string field theory defined on $\mathbb{R}^{1,d} \times T^{\tilde{d}}$, the state can be labelled by

$$|P_I, w^a\rangle := |P_\mu, P_a, w^a\rangle$$

where $P_I = (P_\mu, P_a)$ is the momentum and w^a is the winding number. By imposing the periodic conditions $X^a \sim X^a + 2\pi$, the eigenvalues of P_a and w^a will take the integer numbers.

The general state $|\Psi\rangle$ takes the form

$$|\Psi\rangle = \sum_m \int dP_\mu \sum_{P_a} \sum_{W_a} \Psi_m(P_\mu, P_a, W_a)\, \mathcal{O}^m\, |P_\mu, P_a, W^a\rangle$$

where $\mathcal{O}^m$ are made of the matter and ghost oscillators and $\Psi_m(P_\mu, P_a, W_a)$ are fields in the momentum space.

Like X^μ and X^a are conjugate to the momentum P_μ and P_a, one can introduce a new coordinate $\tilde{X}_a$ which is conjugate to the winding number W^a. Then, in the coordinate space, the fields are

$$\Psi_m(X^\mu, X^a, \tilde{X}_a)$$

where X^a and $\tilde{X}_a$ are periodic on the doubled torus $T^{2\tilde{d}}$.

The $D = d + \tilde{d}$ dimensional mass squared is

$$M_D^2 = \frac{2}{\alpha'}(N_L + N_R - 2)$$

while the mass squared for the non-compact d-dimensional $\mathbb{R}^{1,d}$ is given by

$$m_d^2 = \frac{2}{\alpha'}(N_L + N_R - 2) + (P_a)^2 + (W_a)^2$$

For the level $(N_L, N_R) = (1,1)$ with the mass $M_D^2 = 0$,

the fields are

$$h_{MN}(X^\mu, X^a, \tilde{X}_a), \quad B_{MN}(X^\mu, X^a, \tilde{X}_a), \quad \Phi_{MN}(X^\mu, X^a, \tilde{X}_a)$$

which are the graviton field, Kalb-Ramond field (rank-2 antisymmetric field) and dilaton field respectively. Since they are all depending on the X and $\tilde{X}$, for the quadratic theory, the linearized gauge (double diffeomorphism) transformations have the form

$$\delta h_{MN} = \partial_M \xi_N + \partial_N \xi_M + \tilde{\partial}_M \tilde{\xi}_N + \tilde{\partial}_N \tilde{\xi}_M$$

$$\delta B_{MN} = -(\tilde{\partial}_M \xi_N - \tilde{\partial}_N \xi_M) - (\partial_M \tilde{\xi}_N - \partial_N \tilde{\xi}_M)$$

$$\delta \Phi = -\frac{1}{2} \partial \cdot \xi + \frac{1}{2} \tilde{\partial} \cdot \tilde{\xi}$$

where $\tilde{X}_M = (\tilde{X}_a, 0)$ and $\tilde{\partial}_M = (\partial / \partial \tilde{X}_a, 0)$

Consider the Einstein-Hilbert action (ignore the constant factor)

$$S = \int \sqrt{-g}\, R$$

to the quadratic order, the action in terms of the fluctuation field $h_{MN}(X) = g_{MN}(X) - \eta_{MN}$ is

$$S_0 = \int dX \left[\frac{1}{4} h^{MN} \partial^2 h_{MN} - \frac{1}{4} h \partial^2 h + \frac{1}{2} (\partial^M h_{MN})^2 + \frac{1}{2} h \partial_M \partial_N h^{MN} \right]$$

For the field $h_{MN}(X,\tilde{X})$ depending on X and $\tilde{X}$,

$$S=\int dX\,d\tilde{X}\Big[\tfrac{1}{4}h^{MN}\partial^2 h_{MN}-\tfrac{1}{4}h\partial^2 h+\tfrac{1}{2}(\partial^M h_{MN})^2+\tfrac{1}{2}h\partial_M\partial_N h^{MN}$$
$$+\tfrac{1}{4}h^{MN}\tilde{\partial}^2 h_{MN}-\tfrac{1}{4}h\tilde{\partial}^2 h+\tfrac{1}{2}(\tilde{\partial}^M h_{MN})^2+\tfrac{1}{2}h\tilde{\partial}_M\tilde{\partial}_N h^{MN}\Big]$$

If requiring the double diffeomorphisms for $h_{MN}(X,\tilde{X})$ as

$$\delta h_{MN}=\partial_M\xi_N+\partial_N\xi_M \quad \text{and} \quad \tilde{\delta}h_{MN}=\tilde{\partial}_M\tilde{\xi}_N+\tilde{\partial}_N\tilde{\xi}_M$$

then

$$\tilde{\delta}S=\int dX\,d\tilde{X}\Big[h_{MN}\partial^2\tilde{\partial}_M\tilde{\xi}_N+\partial_M h^{MN}(\partial^K\tilde{\partial}_K)\tilde{\xi}_N-h\partial^2\tilde{\partial}\cdot\tilde{\xi}$$
$$+h(\partial_M\tilde{\partial}^M)\partial_N\tilde{\xi}^N+\partial_M h^{MN}\partial^K\tilde{\partial}_N\tilde{\xi}_K+(\partial_M\partial_N h^{MN})\tilde{\partial}\cdot\tilde{\xi}\Big]$$

$$\Rightarrow \tilde{\delta}S=\int dX\,d\tilde{X}\Big[h^{MN}\partial^2\tilde{\partial}_M\tilde{\xi}_N-h^{MN}\partial_M\partial^K\tilde{\partial}_N\tilde{\xi}_K$$
$$+(\partial_M\partial_N h^{MN}-\partial^2 h)\tilde{\partial}\cdot\tilde{\xi}+(\partial^M h_{MN}-\partial_N h)(\partial\cdot\tilde{\partial})\tilde{\xi}^N\Big]$$

$$\Rightarrow \tilde{\delta}S=\int dX\,d\tilde{X}\Big[(\tilde{\partial}_N h^{MN})\partial^K(\partial_M\tilde{\xi}_K-\partial_K\tilde{\xi}_M)$$
$$+(\partial_M\partial_N h^{MN}-\partial^2 h)\tilde{\partial}\cdot\tilde{\xi}$$
$$+(\partial^M h_{MN}-\partial_N h)(\partial\cdot\tilde{\partial})\tilde{\xi}^N\Big] \qquad (*)$$

where assuming $\tilde{\xi}$ satisfies the constraint $(\partial\cdot\tilde{\partial})\tilde{\xi}=0$.

To cancel $\tilde{\delta} S$, new terms added into the action are

$S_1 = \int dX d\tilde{X} (\tilde{\partial}_N h^{MN}) \partial^K B_{MK}$ with $\tilde{\delta} B_{MN} = -(\partial_M \tilde{\xi}_N - \partial_N \tilde{\xi}_M)$

and

$S_2 = \int dX d\tilde{X} [-2(\partial_M \partial_N h^{MN} - \partial^2 h) \Phi]$ with $\tilde{\delta} \Phi = \frac{1}{2} \tilde{\partial} \cdot \tilde{\xi}$

Naturely, the Kalb-Ramond field and dilaton field are required by the gauge symmetry and $\tilde{\xi}$ satisfies the constraint $\partial \cdot \tilde{\partial} = 0$.

References:

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