When are volume and spinors possible?

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Outline

- Orientability
- Čech cohomology groups
- Stiefel–Whitney classes
- The “first” Stiefel–Whitney Theorem
- Spin Structure
- The “Second” Stiefel–Whitney Theorem
- Examples

Orientability

Let U_i be an open set over Manifold, $\mathcal{M}$,

where each U_i have their own coordinates

$$U_i \cap U_j \cap U_k \neq \emptyset$$

t_{ij} jacobian from U_j to U_i

$$t_{ii} = \mathbb{I} \qquad t_{ij}t_{jk}t_{kj} = \mathbb{I}$$

Čech cohomology groups

Multiplicative Group $\mathbb{Z}_2 = \{-1, 1\}$

Čech r-cochain $:= f(i_1 \ldots i_r) \in \mathbb{Z}_2$

defined on $U_{i_1} \cap \ldots \cap U_{i_r} \neq \emptyset$

f is also defined to be totally symmetric!

Čech r-cochains (coboundary operator)

For C^r be the multiplicative groups of Čech r-cochains

Define the coboundary operator,

$\delta : C^r \rightarrow C^{r+1}$

$(\delta f) := \prod_{j=0}^{r+1} f(i_i, \ldots, \hat{i}_j, \ldots, i_{r+1})$

Note: $\delta^2 f \equiv 1$(nilpotent)

Čech r-cochains (coboundary operator)

r[th] - Cocycle group $\equiv Z^r := \{f \in C^r | \delta f = 1\}$

r[th] - Coboundary group $\equiv B^r := \{f \in C^r | \exists f' \in C^{r-1} s.t. \delta f' = f\}$

r[th] - Cohomology group $\equiv H^r := \ker \delta_r / \mathrm{im}\, \delta_{r-1} \equiv Z^r / B^r$

The “first” Stiefel–Whitney Theorem

With metric (positive def.) g on $\mathcal{M}$

t_{ij} can be restricted to $t_{ij} \in O(m)$

With a change of frame trans, $h_i \in O(m)$,

defined in U_i

The “first” Stiefel–Whitney Theorem

$$f_0(i) := \det(h_i)$$

$$f_1(i, j) := \det(t_{ij})$$

The “first” Stiefel–Whitney Theorem

The “first” Stiefel–Whitney Theorem: Let **TM → M** be a tangent bundle with **fibre metric**. M is **orientable if and only if** the **First Stiefel-Whitney Class** is **trivial**.

The first Stiefel-Whitney Class is a obstruction to **orientability**.

Spin Structure(Now assume orientability)

Spin Structure $:=$ a 2 to 1 homomorphism

$$\phi : SPIN(m) \to SO(m)$$

$$\tilde{t}_{ij} \in SPIN(m)$$

$$\phi(\tilde{t}_{ij}) = t_{ij}$$

it can be shown that:

$$\phi(\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki}) = t_{ij}t_{jk}t_{ki} = \mathbb{I}$$

Spin Structure (Now assume orientability)

$$f_1(i,j) := \{x \in \{\pm 1\} | x\mathbb{I} \in \phi^{-1}(t_{ij})\}$$

$$f_2(i,j,k) := \{x \in \{\pm 1\} | \tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = x\mathbb{I}\}$$

The “Second” Stiefel–Whitney Theorem

The “second” Stiefel–Whitney Theorem: Let **TM be the tangent bundle** over an **orientable manifold M**. There exists a *spin bundle(spinors)* over *M* if and only if the **second whitney class** is trivial.

Examples

$$w_1(\mathbb{R}P^m) = x$$

$$w_1(\text{Klein bottle}) = x$$

$$w_1(\mathbb{C}P^m) = 1$$

$$w_2(\mathbb{C}P^m) = \begin{cases} x & \text{m odd} \\ 0 & \text{m even} \end{cases}$$

$$w_1(S^m) = w_2(S^m) = 1$$

$$w_1(\Sigma_g^m) = w_2(\Sigma_g^m) = 1$$

LINKS and Thanks!

Theoretical and Mathematical Physics Part II. Fibre Bundles, Topology and Gauge Fields, Gerd Rudolph Matthias Schmidt

Geometry, Topology and Physics (Graduate Student Series in Physics), Mikio Nakamura

https://math.stackexchange.com/questions/808263/spin-manifold-and-the-second-stiefel-whitney-class/808396#808396

https://ncatlab.org/nlab/show/spin+structure

https://www.iri.upc.edu/people/ros/StructuralTopology/ST17/st17-05-a2-ocr.pdf