Spinor and Twistor Basics.

We know that, such as, ~~$u_-(\vec{p})$~~ the Spinor $u_-(\vec{p})$ could be writen as:

$$u_-(\vec{p}) = \begin{pmatrix} \phi_a \\ 0 \end{pmatrix}$$

and ϕ_a is a two-component numerical spinor. It is not an anticommuting object, such a commuting spinor is sometimes called a twistor. We know that

$$u_s(\vec{p})\bar{u}_s(\vec{p}) = \frac{1}{2}(1+s\gamma_5)(-\not{p}), \quad \text{so at here}$$

$$u_-(\vec{p})\bar{u}_-(\vec{p}) = \frac{1}{2}(1-\gamma_5)(-\not{p}). \quad \text{If we define}$$

$p_{a\dot{a}} \equiv p_\mu \sigma^\mu_{a\dot{a}}$, and also

$p^{\dot{a}a} = \varepsilon^{ac}\varepsilon^{\dot{a}\dot{c}} p_{c\dot{c}} = p_\mu \bar{\sigma}^{\mu\dot{a}a}$, we could have

$$u_-(\vec{p})\bar{u}_-(\vec{p}) = \begin{pmatrix} 0 & -p_{a\dot{a}} \\ 0 & 0 \end{pmatrix} \quad \text{Also}$$

$$u_-(\vec{p})\bar{u}_-(\vec{p}) = \begin{pmatrix} 0 & \phi_a \phi^*_{\dot{a}} \\ 0 & 0 \end{pmatrix} \quad \text{w/ } \phi^*_{\dot{a}} = (\phi_a)^*$$

So: $p_{a\dot{a}} = -\phi_a \phi^*_{\dot{a}}$

~~Th~~ Twistor theory was first introduced about fifty years ago. Despite many interesting initial advances, the subject stalled significantly by the late 1980s due to a variety technical and philosophical problems. It was resurrected for physics in 2003 that twistor theory can be combined with string perturbation theory to calculate the entire tree-level S-matrix of Yang-Mills theory in four — space - time dimensions.

We start with a flat, four-dimensional Minkowski space time, M, with signature $(+,-,-,-)$.

Metric signature; In mathematics, the signature (v, p, r) of a metric tensor g is the number of positive, negative and zero eigenvalues of the real symmetric matrix g_{ab} of the metric tensor with respect to a basis. In physics, the v represents time or virtual dimension, and the p for the space and physical dimension. By Sylvester's law of inertia, these numbers do not depend on the choice of basis. The signature thus classifies the metric up to a choice of basis. The signature is often denoted by a pair of integers (v, p) implying $r=0$, or as an explicit list of signs eigen values such as $(+,-,-,-)$ or $(-,+,+,+)$ for the signatures $(1,3,0)$ and $(3,1,0)$

*. Complexified Minkowski space

Let $\mathcal{M}$ be a real, d-dimensional space-time with a metric $ds^2 = g_{ab}(x)\, dx^a dx^b$ in some coordinate system x^a. The complexification of $(\mathcal{M}, g_{ab})$ means x^a could take complex values and extending $g_{ab}(x)$ holomorphically. (By holomorphic, we mean there is no $\bar{x}^a$-dependence in the metric after complexification.) The resulting complexified space-time is denoted $\mathcal{M}_{\mathbb{C}}$.

For four-dimensional Minkowski space-time M, ~~in~~ in Cartesian coordinates $x^a = (x^0, x^1, x^2, x^3)$ with metric $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$.

Complexified Minkowski space $M_{\mathbb{C}}$ is just $\mathbb{C}^4$, η_{ab} doesn't change.
The line element

$$ds^2 = \eta_{ab}\, dx^a dx^b = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

looks the same as in real Minkowski space, just x^0, x^1, x^2, x^3 now could take complex values.

But the signature of this complexified metric is no longer meaningful. real flat space of any signature can be obtained by taking different real slices of the complexified space-time. Such as, the real slice ~~of~~ of real Minkowski space-time, $M \subset M_{\mathbb{C}}$. By taking different real slices we can obtain $\mathbb{R}^4$ with Euclidean signature $(+,+,+,+)$ or $\mathbb{R}^{2,2}$ with split signature $(+,+,-,-)$

Euclidean: $\mathbb{R}^4 \subset M_{\mathbb{C}}$, $x^0 \in \mathbb{R}$, $x^1, x^2, x^3 \in i\mathbb{R}$

split: $\mathbb{R}^{2,2} \subset M_{\mathbb{C}}$, $x^0, x^2, x^3 \in \mathbb{R}$, $x^1 \in i\mathbb{R}$

$$\therefore\ ds^2_{Eu} = +x^{0^2} + x^{1^2} + x^{2^2} + x^{3^2} = x^{0^2} - |x^1|^2 - |x^2|^2 - |x^3|^2$$

$$ds^2_{sp} = +x^{0^2} + x^{1^2} - x^{2^2} - x^{3^2} = x^{0^2} - |x^1|^2 - x^{2^2} - x^{3^2}$$

In this sense, complexified Minkowski space is a sort of universal analytic continuation of all flat, real space-times. Complexification means we can study physics on $M_{\mathbb{C}}$, then recover results in the desired space-time signature by imposing appropriate reality conditions later.

*. 2-spinors in Minkowski space.

The spin group of complexified Minkowski space is $SO(4,\mathbb{C})$. ③

which is locally isomorphic to $SL(2,C) \times SL(2,C)$. That is, the Lie algebra $so(4,C)$ is isomorphic to ~~$sl(2)$~~ $SL(2,C) \times SL(2,C)$.

A vector on M_C lives in the $(\frac{1}{2},\frac{1}{2})$ representation of $SL(2,C) \times SL(2,C)$, so any vector index can be represented by a pair of $SL(2,C)$ indices: one in the $(\frac{1}{2},0)$ representation and the other in the $(0,\frac{1}{2})$ representation.

The equivalence between a vector index on M_C and two conjugate $SL(2,C)$ spinor indices is ~~nothing~~ given by Pauli matrices, σ_a. For a vector $v^a = (v^0, v^1, v^2, v^3)$, it's representation in terms of $SL(2,C)$ Weyl spinors is:

$$v^{\alpha\dot{\alpha}} := \frac{\sigma_a^{\alpha\dot{\alpha}}}{\sqrt{2}} v^a = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0+v^3 & v^1-iv^2 \\ v^1+iv^2 & v^0-v^3 \end{pmatrix}$$

The un-dotted spinor indices $(\alpha=0,1)$ lives in the $(\frac{1}{2},0)$ representation of $SL(2,C) \times SL(2,C)$, and will be refered to as negative chirality spinor indices. The dotted spinor indices $(\dot{\alpha}=\dot{0},\dot{1})$ lives in the $(0,\frac{1}{2})$ representation and will be referred to as positive chirality spinor indices.

This rule (contracting with the Pauli matrices) allows us to replace any number of vector indices on M_C with pairs of spinor indices. For example,

$$T^{abc} \rightarrow T^{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}}$$

From it, we could calculate the norm of a vector v^a as:

$$\eta_{ab} v^a v^b = 2\det(v^{\alpha\dot{\alpha}})$$

This means v^a is null if and only if $\det(v^{\alpha\dot{\alpha}})$ vanishes. But $v^{\alpha\dot{\alpha}}$ is 2×2, so its determinant vanishes if and only if it's rank is less than 2. Therefore, every non-trivial null vector in $M_{\mathbb{C}}$ can be written

$$v^{\alpha\dot{\alpha}}_{\text{null}} = a^{\alpha}\tilde{a}^{\dot{\alpha}}$$

for some spinors a^{α}, $\tilde{a}^{\dot{\alpha}}$. The converse is also true. That is, any matrix of the form $a^{\alpha}\tilde{a}^{\dot{\alpha}}$ has vanishing determinant, and hence it's corresponding vector is null.

So, any pair of Weyl spinors of opposite chirality define a null vector.

Also, we could raise and lower spinor indices by the ~~two~~-dimensional Levi-Civita symbols:

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon_{\dot{\alpha}\dot{\beta}}$$

They are skew-symmetric

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$$

and their inverses are defined by:

$$\varepsilon^{\alpha\beta}\varepsilon_{\gamma\beta} = \delta^{\alpha}_{\gamma} \qquad \varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta} = 2$$

And likewise for dotted indices.

There is a convention for how to raise and lower spinor indices. It is 'lower to the right, raise to the left':

$$a_{\alpha} := a^{\beta}\varepsilon_{\beta\alpha}, \quad b^{\alpha} := \varepsilon^{\alpha\beta}b_{\beta}$$

with identical conventions for dotted spinor indices.

So given some vector $v^{\alpha\dot{\alpha}}$ (in spinor representation),

$$v_{\alpha\dot{\alpha}} = v^{\beta\dot{\beta}} \varepsilon_{\beta\alpha} \varepsilon_{\dot{\beta}\dot{\alpha}}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 - v^3 & -(v^1 + i v^2) \\ -v^1 + i v^2 & v^0 + v^3 \end{pmatrix}$$

It is easy to see that

$$v^{\alpha\dot{\alpha}} v_{\alpha\dot{\alpha}} = 2\det(v^{\alpha\dot{\alpha}}) = \eta_{ab} v^a v^b$$

In the 2-spinor formalism the line element for M_c is

$$ds^2 = \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} dx^{\alpha\dot{\alpha}} dx^{\beta\dot{\beta}}$$

By the use of Levi-Civita symbols, we could define inner products on the space of negative and positive chirality spinors.

$$\langle \kappa \omega \rangle := \kappa^{\alpha} \omega_{\alpha} = \kappa^{\alpha} \omega^{\beta} \varepsilon_{\beta\alpha}$$

$$[\tilde{\kappa} \tilde{\omega}] := \tilde{\kappa}^{\dot{\alpha}} \tilde{\omega}_{\dot{\alpha}} = \tilde{\kappa}^{\dot{\alpha}} \tilde{\omega}^{\dot{\beta}} \varepsilon_{\dot{\beta}\dot{\alpha}}$$

These are the natural $SL(2,\mathbb{C})$-invariant, skew-symmetric inner products on the 2-spinors of each chirality.

For example, consider any two null vectors v^a_{null} and w^a_{null} in M_c, ~~as we~~ these can be written as $v^a_{null} \leftrightarrow \kappa^{\alpha} \tilde{\kappa}^{\dot{\alpha}}$ and $w^a_{null} \leftrightarrow \omega^{\alpha} \tilde{\omega}^{\dot{\alpha}}$ for some spinors $\{\kappa^{\alpha}, \tilde{\kappa}^{\dot{\alpha}}, \omega^{\alpha}, \tilde{\omega}^{\dot{\alpha}}\}$. The inner product:

$$v_{null} \cdot w_{null} = \langle \kappa \omega \rangle [\tilde{\kappa} \tilde{\omega}]$$

- note: Skew-symmetric matrix.

In mathematics, particularly in linear algebra, a skew-symmetric (or antisymmetric or antimetric) matrix is a square matrix whose transpose equals its negative

$$A \text{ skew-symmetric} \leftrightarrow A^T = -A$$

$$a_{ji} = -a_{ij}$$

* Real slices and spinor conjugations

We have translated $M_{\mathbb{C}}$ into 2-spinors, now consider how real slices of various signature can be singled out at the level of the spinor formalism. That is, find reality conditions on the matrix

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix}$$

1. Lorentzian signature

We try to single out the usual, Lorentz real Minkowski space M inside of $M_{\mathbb{C}}$.

Requiring $x^{\alpha\dot{\alpha}} = (x^{\alpha\dot{\alpha}})^{\dagger}$, where

$$(x^{\alpha\dot{\alpha}})^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{x}^0+\bar{x}^3 & \bar{x}^1-i\bar{x}^2 \\ \bar{x}^1+i\bar{x}^2 & \bar{x}^0-\bar{x}^3 \end{pmatrix}$$

In fact, ~~since~~ $(x^{\alpha\dot{\alpha}})^{\dagger}$ has the same form as $x^{\alpha\dot{\alpha}}$, just x^a is replaced with $\bar{x}^a$, so if (x^0, x^1, x^2, x^3) is ~~are~~ real, it should ~~have~~ fit this relation.

But Hermitian conjugation includes the transpose operation, during this process the positive and negative chirality spinor representations are exchanged. Thus, the reality structure associated with the Lorentzian-real slice of $M_{\mathbb{C}}$ is naturally associated with a complex conjugation on 2-spinors which exchanges dotted and un-dotted spinors. So, given spinors with components $\kappa^{\alpha} = (a, b)$ and ~~$\tilde{\omega}$~~ $\tilde{\omega}^{\dot{\alpha}} = (c, d)$ with $a, b, c, d \in \mathbb{C}$, this conjugation operation acts as:

$$\kappa^{\alpha} \to \bar{\kappa}^{\dot{\alpha}} = (\bar{a}, \bar{b}), \quad \tilde{\omega}^{\dot{\alpha}} \to \bar{\tilde{\omega}}^{\alpha} = (\bar{c}, \bar{d})$$

We could show that any real null vector in M can be written as $\kappa^\alpha \bar{\kappa}^{\dot\alpha}$ for some spinor κ^α.

② Euclidean signature

Define $\hat{x}^{\alpha\dot\alpha} := \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{x}^0 - \bar{x}^3 & -\bar{x}^1 + i\bar{x}^2 \\ -\bar{x}^1 - i\bar{x}^2 & \bar{x}^0 + \bar{x}^3 \end{pmatrix}$

and demand $x^{\alpha\dot\alpha}$ be preserved under this operation. That is,

$$x^{\alpha\dot\alpha} = \hat{x}^{\alpha\dot\alpha}$$

$$x^{\alpha\dot\alpha}\big|_{x=\hat{x}} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + iy^3 & iy^1 + y^2 \\ -y^1 - y^2 & x^0 - iy^3 \end{pmatrix}, \quad x^0, y^1, y^2, y^3 \in \mathbb{R}$$

For example, let $\bar{x}^0 - \bar{x}^3 = x^0 + x^3 \Rightarrow \underbrace{\bar{x}^0 - x^0}_{\text{imagine}} = \underbrace{\bar{x}^3 + x^3}_{\text{Real}} = 0$

So x^0 has no imagine part; x^0 is real and x^3 is imagine.

We could also see that $x^2 = 2\det(x) = (x^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2$

This 'hat-operation' induces a ~~non~~ conjugation on 2-spinors which does not change spinor representations:

$$\kappa^\alpha \to \hat{\kappa}^\alpha = (-\bar{b}, \bar{a}), \quad \tilde{\omega}^{\dot\alpha} \to \hat{\omega}^{\dot\alpha} = (-\bar{d}, \bar{c})$$

Note that $\hat{\hat{\kappa}}^\alpha = -\kappa^\alpha$, we need to apply the hat-conjugation four times to get back to the spinor we started from. So the reality structure associated with Euclidean signature is often referred to as quaternionic.

We could see that there is no non-trivial combination $\kappa^\alpha \tilde{\omega}^{\dot\alpha}$ which is preserved under the hat-operation.

This is the statement that there are no real null vectors in Euclidean space.

③ Split signature.

Take the complex conjugate of $x^{\alpha\dot\alpha}$

$$\overline{x^{\alpha\dot\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{x}^0+\bar{x}^3 & \bar{x}^1+i\bar{x}^2 \\ \bar{x}^1-i\bar{x}^2 & \bar{x}^0-\bar{x}^3 \end{pmatrix}$$

and ask for $x^{\alpha\dot\alpha} = \overline{x^{\alpha\dot\alpha}}$. This force

$$x^{\alpha\dot\alpha}\Big|_{x=\bar x} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0+x^3 & x^1+y^2 \\ x^1-y^2 & x^0-x^3 \end{pmatrix} \qquad x^0, x^1, y^2, x^3 \in R$$

For example, $\bar{x}^1 - i\bar{x}^2 = x^1 + ix^2 \Rightarrow$

$$\underbrace{\bar{x}^1 - x^1}_{\text{image}} = i\underbrace{(\bar{x}^2+x^2)}_{\text{Real}} = 0$$

$\therefore x^1$ is real and x^2 is image.

For this $x^{\alpha\dot\alpha}$, $x^2 = 2\det(x) = (x^0)^2 + (y^2)^2 - (x^1)^2 - (x^3)^2$

The underlying conjugation on 2-spinors is ordinary complex conjugation, it does not interchange the spinor representation, So ~~the~~ in split signature the conjugation acts on spinors as:

$$\kappa^\alpha \to \overline{\kappa^\alpha} = (\bar a, \bar b) \quad \text{and} \quad \tilde{w}^{\dot\alpha} \to \overline{\tilde{w}^{\dot\alpha}} = (\bar c, \bar d)$$

*- Twistor space.

Let ~~CP³~~ CP^3 be the 3-dim. complex projective space, it is the space of all complex lines through the origin in C^4. Also, CP^3 can be described as homogeneous coordinates $Z^A = (Z^1, Z^2, Z^3, Z^4)$, which take values in the complex numbers, but are never all vanishing, and are identified up to overall re-scalings:

$$(Z^1, Z^2, Z^3, Z^4) \neq (0,0,0,0), \quad rZ^A \sim Z^A, \ \forall r \in C^*$$

C^* is the set of all non-zero complex numbers. The C^* rescalings also called projective rescalings, the invariance under it means the homogeneous coordinates only contain three complex degrees of freedom. We can chart CP^3 by covering it with the coordinate patches $U_i = \{Z^A \in C^4 \mid Z^i \neq 0\}$, in U_i there are three complex coordinates given by taking $(Z^i)^{-1} Z^A$.

The twistor space PT of complexified Minkowski space is defined to be an open subset of the complex projective space CP^3. On PT it is useful to divide the four homogeneous coordinates Z^A into two Weyl spinors of opposite chirality

$$Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$$

We need to define a relationship between PT and space-time. It is non-local and is referred to as the twistor correspondence.

$$\mu^{\dot\alpha} = x^{\alpha\dot\alpha} \lambda_\alpha$$

These are known as the incidence relations.

This relationship is often presented in terms of a double fibration of the projective spinor bundle over M_c and PT

$$\begin{tikzcd} & PS \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ PT & & M_c \end{tikzcd}$$

PS has coordinates $(x^{\alpha\dot\alpha}, \lambda_\beta)$, with $\lambda_\beta \sim r\lambda_\beta$ for all non-zero complex numbers r. So, at here, the spinor λ_β acts as a homogeneous coordinate on the one-dimensional complex projective space CP^1, a Riemann sphere. So $PS \cong M_C \times CP^1$, and $\pi_1 : PS \to M_c$ is $(x^{\alpha\dot\alpha}, \lambda_\beta) \to x^{\alpha\dot\alpha}$ while $\pi_2 : PS \to PT$ is $(x^{\alpha\dot\alpha}, \lambda_\beta) \to (x^{\beta\dot\alpha}\lambda_\beta, \lambda_\alpha)$

The relation
$$\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$$

If we don't consider ~~the~~ coefficient r, then ~~relation~~ $Z_A =$ $(\mu^{\dot\alpha}, \lambda_\alpha)$ are just coordinates on C^4, so this relation defines a complex plane $C^2 \subset C^4$. Then consider the r, ~~the~~ this relation defines a $CP^1 \subset PT$, which is a Riemann sphere. So a point in M_C corresponds to a linearly and holomorphically (no complex conjugations appear any where) embedded Riemann sphere in twistor space.

More precisely, any holomorphic linear embedding of a Riemann sphere into CP^3 can always be put into this form. If $\sigma_a = (\sigma_0, \sigma_1)$ are homogeneous coordinates on CP^1, we could have
$$\mu^{\dot\alpha} = b^{\dot\alpha a}\sigma_a, \quad \lambda_\alpha = c^a_\alpha \sigma_a$$

For $(b^{\dot\alpha a}, c^a_\alpha)$, there are 8 complex parameters, but they are not independent. There are 4 conditions, 3 from the automorphisms of CP^1 which are the Möbius transformation, and 1 for the C^* projective rescalings, those can be used to fix $c^a_\alpha = \delta^a_\alpha$. So, $\mu^{\dot\alpha} = b^{\dot\alpha a}\sigma_a$, $\lambda_\alpha = \delta^a_\alpha \sigma_a$

Möbius transformation is a projection from Riemman sphere to itself.

$$f(z) = \frac{az+b}{cz+d} \quad , \text{ with } ad-bc \neq 0$$

it defines on $\hat{C} = C \cup \{\infty\}$, the complex plane plus the infinity. it is also the Riemann sphere, So Möbius ~~trans~~ transformation transforms Riemann sphere to itself.

Given 3 different points z_1, z_2 and z_3, there is only one Möbius transformation f, $f(z_1) = 0$, $f(z_2) = 1$, $f(z_3) = \infty$.

The upshot of this is that a point in Minkowski space corresponds to a holomorphically linearly embeded Riemann sphere in twistor space. For $x \in M_C$, we denote the corresponding Riemann sphare in twistor space by $X \cong CP^1 CPT$. And those Riemann sphere are always referred to as lines. (The line X associated to $x \in M_C$), So now a point in space-time is described by an extended object in twistor space.

Inversly, we could fine ~~how~~ what a point in twistor space corresponds to in space-time, Suppose we have a point $Z \in PT$ as the intersection of two lines X and Y,

$$X \cap Y = \{Z \in PT\} \Rightarrow \begin{cases} \mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \lambda_\alpha \\ \mu^{\dot{\alpha}} = y^{\alpha\dot{\alpha}} \lambda_\alpha \end{cases}$$

for two points $x, y \in M_C$. So

$$(x-y)^{\alpha\dot{\alpha}} \lambda_\alpha = 0$$

If $(x-y)^{\alpha\dot{\alpha}} \neq 0$, then only $(x-y)^{\alpha\dot{\alpha}} \propto \lambda^{\alpha}$ could fix it.

$$(x-y)^{\alpha\dot{\alpha}} \lambda_{\alpha} = \epsilon_{\alpha\beta}(x-y)^{\alpha\dot{\alpha}} \lambda^{\beta}$$

Therefore, the lines X, Y in twistor space intersect in a point Z if and only if

$$(x-y)^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

Just consider $(x-y)^{\alpha\dot{\alpha}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \propto \lambda^{\alpha}$, ~~if (c,d) ≠ (a,b~~

if $(c,d) \neq (a,b)\cdot$ Const. then this condition ~~of~~ can't work.

$\therefore$ $x, y \in M_C$ are null separated.

That is, lines in twistor space intersect if and only if their corresponding points in M_C are null separated.

Spacetime