An Introduction To IKKT Matrix Model

1. Green-Schwarz superstring action

$$S_{GS} = S_1 + S_2$$

$$S_1 = -T\int d^2\sigma \sqrt{-\det(\Pi^{\mu}_{\alpha}\Pi_{\beta\mu})}\ , \quad T = \frac{1}{\lambda} \text{ is the string tension}$$

$$\Pi^{\mu}_{\alpha} = \partial_\alpha X^\mu - \bar{\theta}^A \Gamma^\mu \partial_\alpha \theta^A, \quad A = 1, 2, \cdots, N.$$

where Π^{μ}_{α} is the supersymmetric generalization of $\partial_\alpha X^\mu$ part in the Nambu-Goto bosonic string action, which is

$$S_{NG} = -T\int d^2\sigma \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)}$$

For the type IIB superstring, $D=10$, $N=2$. Therefore, θ^{Aa} are Majorana-Weyl spinors with $A=1,2$ and $a = 1, 2, \cdots, 32$ (where $2^{[\frac{D}{2}]} = 2^5 = 32$). However, the independent components of θ^{1a}, θ^{2a} are 16 real components. Also, θ^1 and θ^2 have the same chiralities.

Make an analytic continuation $\theta^2 \to i\theta^2$, then

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - i\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1 + i\bar{\theta}^2\Gamma^\mu\partial_\alpha\theta^2$$

The second term S_2 is required by the K-symmetry

$$S_2 = i\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 + \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2) + \epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2$$

The S_{GS} has the $N=2$ spacetime supersymmetry

$$\delta\theta^A = \epsilon^A, \quad \delta\bar{\theta}^A = \bar{\epsilon}^A, \quad \delta X^\mu = i\bar{\epsilon}^1\Gamma^\mu\theta^1 - i\bar{\epsilon}^2\Gamma^\mu\theta^2$$

here ϵ^A is a constant infinitesimal Grassmann parameter, which is a τ and σ - independent Majorana-Weyl spinor.

S_{GS} also has the K-symmetry

$$\delta_K\theta^1 = \alpha^1, \quad \delta_K\theta^2 = \alpha^2,$$

$$\delta_K X^\mu = i\bar{\theta}^1\Gamma^\mu\alpha^1 - i\bar{\theta}^2\Gamma^\mu\alpha^2$$

where $\alpha^1 = (1+\tilde{\Gamma})K^1$, $\alpha^2 = (1-\tilde{\Gamma})K^2$, $\tilde{\Gamma} = \frac{1}{2\sqrt{-\frac{1}{2}\Sigma^2}}\Sigma_{\mu\nu}\Gamma^{\mu\nu}$

and $\Sigma^{\mu\nu} = \epsilon^{\alpha\beta}\Pi^\mu_\alpha\Pi^\nu_\beta$, $\Gamma^{\mu\nu} = \Gamma^{[\mu}\Gamma^{\nu]}$

The K-symmetry can be gauge fixing by setting

$$\theta^1 = \theta^2 = \psi$$

The gauge fixed $\tilde{S}_{GS}$ is

$$\tilde{S}_{GS} = -T\int d^2\sigma \left(\sqrt{-\frac{1}{2}\tilde{\Sigma}^2} + 2i\epsilon^{\alpha\beta}\partial_\alpha X^\mu \bar{\psi}\Gamma_\mu \partial_\beta \psi\right)$$

where $\tilde{\Sigma}^{\mu\nu} = \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu$

The $\tilde{S}_{GS}$ has the new $N=2$ supersymmetry

$$\tilde{\delta}\theta^1 = \delta\theta^1 + \delta_K\theta^1, \quad \tilde{\delta}\theta^2 = \delta\theta^2 + \delta_K\theta^2$$

$$\tilde{\delta}X^\mu = \delta X^\mu + \delta_K X^\mu$$

with $K^1 = \frac{1}{2}(-\epsilon^1 + \epsilon^2)$ and $K^2 = \frac{1}{2}(\epsilon^1 - \epsilon^2)$, $\delta\theta^1 = \delta\theta^2$

By defining $\xi = \frac{1}{2}(\epsilon^1 + \epsilon^2)$, $\epsilon = \frac{1}{2}(\epsilon^1 - \epsilon^2)$, the new $N=2$ supersymmetry can be expressed as

$$\delta^{(1)}\psi = -\frac{1}{2\sqrt{-\frac{1}{2}\tilde{\Sigma}^2}}\tilde{\Sigma}_{\mu\nu}\Gamma^{\mu\nu}\epsilon, \quad \delta^{(1)}X^\mu = 4i\bar{\epsilon}\Gamma^\mu\psi$$

$$\delta^{(2)}\psi = \xi, \quad \delta^{(2)}X^\mu = 0$$

The $\tilde{S}_{GS}$ in the Schild gauge is given by

$$S_{Schild} = \int d^2\sigma \left(\sqrt{g} \left(\frac{1}{4}\{X^\mu, X^\nu\}^2 - \frac{i}{2}\bar{\Psi}\Gamma^\mu\{X^\mu, \Psi\} \right) + \beta\sqrt{g} \right)$$

where $\sqrt{g} = \sqrt{\det g_{\alpha\beta}}$ is from the worldsheet metric $g_{\alpha\beta}$

and $\{X^\mu, Y^\nu\} = \frac{1}{\sqrt{g}} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta Y^\nu$ is the Poisson bracket.

Schild action and $\tilde{S}_{GS}$ are equivalent at least at the classical level. Quantum theory of S_{Schild} is given by the partition function

$$Z_{Schild} = \int D\sqrt{g}\, DX\, D\Psi\, e^{-S_{Schild}}$$

In the classical limit, $\{\ ,\ \} \Rightarrow -i[\ ,\ ]$

$\int d^2\sigma \sqrt{g} \Rightarrow Tr$, then

$$Z = \sum_{n=0}^{\infty} \int dA\, d\Psi\, e^{-S} \quad \text{for large } n \quad \left(\int D\sqrt{g} \Rightarrow \sum_{n=0}^{\infty} \right)$$

where $S = \alpha\left(-\frac{1}{4} Tr[A_\mu, A_\nu]^2 - \frac{1}{2} Tr(\bar{\Psi}\Gamma^\mu [A_\mu, \Psi]) \right) + \beta\, Tr\, \mathbb{1}$

A_μ and Ψ are $n \times n$ hermitian matrices.

The $N=2$ supersymmetry for S is

$$\delta^{(1)}\psi = \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\epsilon, \quad \delta^{(1)}A_\mu = i\epsilon\Gamma_\mu\psi$$

and $\delta^{(2)}\psi = \xi$, $\delta^{(2)}A_\mu = 0$

2. The large N reduction of $\mathcal{N}=1$, $D=10$ supe YM theory

From the Eguchi-Kawai model, for large N limit with $SU(N)$ gauge group, the lattice gauge theory can be replaced the theory defined on only one site, which is independent of the spacetime volume. Then for large N case, it is equivalent to take the dimensional reduction of $D=10$, $\mathcal{N}=1$ SYM to a spacetime point. It is given by

$$S_0 = \frac{Na^4}{g_0^2}\left(-\frac{1}{4}\mathrm{Tr}[A_\mu, A_\nu]^2 - \frac{1}{2}\mathrm{Tr}(\bar{\psi}\Gamma^\mu[A_\mu, \psi])\right)$$

and $-\frac{\pi}{a} \leq$ eigenvalues of $A_\mu < \frac{\pi}{a}$

where a is the spacetime cut-off.

Before the large N reduction, $D=10$ SYM is

$$S_{SYM} = \frac{N}{g_0^2 a^6}\int d^{10}x \left(\frac{1}{4}Tr\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{i}{2}Tr\left(\bar{\psi}\Gamma^{\mu}D_{\mu}\psi\right)\right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$

$$D_\mu \psi = \partial_\mu \psi + i[A_\mu, \psi]$$

Regard the eigenvalues of A_μ as the spacetime coordinates, and β term as the chemical potential, set

$$\alpha = \frac{Na^4}{g_0^2} \sim \frac{1}{\alpha'^2 g_s}, \quad \text{mass scale } m = g_0^{\frac{1}{3}} a$$

then $g_0^2 a^6 = m^6$, $Na^{10} \sim \frac{m^6}{\alpha'^2 g_s}$

So the double scaling limit is

$$a \to 0$$
$$g_0 \sim a^{-3} \to \infty$$
$$N \sim a^{-10} \to \infty$$

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A Large-N Reduced Model as Superstring