

# An Introduction To Mirror Symmetry

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## 1. Mirror Symmetry

For every complex 3-dimensional Calabi-Yau Manifold  $M$ , there associates its mirror manifold  $W$  such that

$$h^{1,1}(M) = h^{1,2}(W)$$

and vice versa, where  $h^{1,1}$  and  $h^{1,2}$  are the Hodge numbers.

The mirror symmetry states that there are two different 2-dimensional  $N=2$  SCFT sigma models such that

$$\text{SCFT}_A(M) \simeq \text{SCFT}_B(W)$$

where " $\simeq$ " means equivalent to one another.

A geometrical realization is given by SYZ (Strominger - Yau - Zaslow) conjecture through the T-duality:

The type IIA string theory ~~is~~ compactified on  $M$  is the same as type IIB string theory compactified on its mirror manifold  $W$ .

2.  $N=(2,2)$  supersymmetry and  $M^{2/2}$  super-Minkowski (2) space

Consider  $N=(2,2)$  susy field theories in 2 dimension.

For  $M^{2/2}$  with flat metric  $\eta = \text{diag}(-1, 1)$ , introduce the bosonic coordinates

$$x^0 = t, \quad x^1 = \sigma$$

as well as the fermionic coordinates (anti-commutative)

$$\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$$

where  $\bar{\theta}$  is the complex conjugation of  $\theta$ .

For a  $N=(2,2)$  susy theory, we need two complex supercharges  $(Q, \bar{Q})$  with positive/negative chiralities respectively. It's ~~not~~ convenient to introduce four real supercharges  $Q_+, \bar{Q}_+, \bar{Q}_-, \bar{Q}_-$  on  $M^{2/2}$

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i \bar{\theta}^{\mp} \partial_{\pm}$$

$$\bar{Q}_{\pm} = - \frac{\partial}{\partial \bar{\theta}^{\pm}} - i \theta^{\pm} \partial_{\pm}$$

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where  $x^\pm = x^0 \pm x^1$  and

$$\partial^\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2} \left( \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right)$$

The supercharges satisfy the anti-commutation relations

$$\{Q^\pm, \bar{Q}^\pm\} = -2i\partial^\pm$$

Under the Lorentz transformation,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \mu, \nu = 0, 1$$

with  $\Lambda = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}$

and  $\theta'^i = S^i{}_j \theta^j, \quad i, j = +, -$

with  $S = \begin{pmatrix} e^{\frac{1}{2}\lambda} & 0 \\ 0 & e^{-\frac{1}{2}\lambda} \end{pmatrix}$

also for  $\bar{\theta}$ ,

$$\bar{\theta}'^i = S^i{}_j \bar{\theta}^j, \quad i, j = +, -$$

### 3. Superfield $\bar{\Phi}(x, \theta, \bar{\theta})$ ④

The superfield is defined as the function on  $M^{2|2}$ .

Since  $(\theta^\pm)^2 = (\bar{\theta}^\pm)^2 = 0$ , then

$$\bar{\Phi}(x, \theta, \bar{\theta}) \equiv \bar{\Phi}(x^0, x^1, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$$

$$= \bar{\Phi}_0(x^0, x^1) + \theta^+ \bar{\Phi}_+(\bar{x}^0, \bar{x}^1) + \theta^- \bar{\Phi}_-(\bar{x}^0, \bar{x}^1)$$

$$+ \bar{\theta}^+ \cancel{\bar{\Phi}_0} \hat{\bar{\Phi}}_+(\bar{x}^0, \bar{x}^1) + \bar{\theta}^- \hat{\bar{\Phi}}_-(\bar{x}^0, \bar{x}^1)$$

$$+ \theta^+ \theta^- \bar{\Phi}_{+-}(\bar{x}^0, \bar{x}^1) + \dots \text{ (up to } 2^4 = 16 \text{ terms)}$$

bosonic superfield :  $[\theta^i, \bar{\Phi}] = 0$ , commutative

fermonic superfield :  $\{ \theta^i, \bar{\Phi} \} = 0$ , anti-commutative

Define

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - i \bar{\theta}^\pm \partial_\pm$$

$$\bar{D}_\pm = - \frac{\partial}{\partial \bar{\theta}^\pm} + i \theta^\pm \bar{\partial}_\pm$$

$$\{ D_\pm, Q_\pm \} = 0 \text{ and } \{ D_\pm, \bar{D}_\pm \} = 2i \partial_\pm$$

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chiral superfield  $\Phi$  :  $\bar{D}_\pm \Phi = 0$

The general form of  $\Phi$  is

$$\Phi(x^\mu, \theta^\pm, \bar{\theta}^\pm) = \phi(y^\pm) + \theta^i \gamma_i(y^\pm) + \theta^+ \theta^- f(y^\pm)$$

where  $y^\pm = x^\pm - i\theta^\pm \bar{\theta}^\pm$ .

anti-chiral superfield :  $D_\pm \bar{\Phi} = 0$

A superfield  $\Phi^t$  is called twisted if

$$\bar{D}_+ \Phi^t = D_- \Phi^t = 0$$

The general form of  $\Phi^t$  is

$$\begin{aligned} \Phi^t(x^\mu, \theta^\pm, \bar{\theta}^\pm) = & \phi^t(\tilde{y}^\pm) + \theta^+ \bar{\chi}_+(\tilde{y}^\pm) + \bar{\theta}^- \chi_-(\tilde{y}^\pm) \\ & + \theta^+ \bar{\theta}^- e(\tilde{y}^\pm) \end{aligned}$$

Similarly, the complex conjugation  $\bar{\Phi}^t$  is a twisted anti-chiral if  $D_+ \bar{\Phi}^t = \bar{D}_- \bar{\Phi}^t = 0$

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#### 4. Chiral superfield theory

Take the  $\theta$ -expansion of  $\Phi$ ,

$$\begin{aligned}\Phi &= \phi(y^\pm) + \theta^i \gamma_i(y^\pm) + \theta^+ \theta^- f(y^\pm) \\ &= \phi - i\theta^+ \bar{\theta}^+ \partial_+ \phi - i\theta^- \bar{\theta}^- \partial_- \phi - \theta^+ \theta^- \bar{\theta}^- \bar{\theta}^+ \partial_+ \partial_- \phi \\ &\quad + \theta^+ \gamma_+ - i\theta^+ \theta^- \bar{\theta}^- \partial_- \gamma_+ + \theta^- \gamma_- - i\theta^- \theta^+ \bar{\theta}^+ \partial_+ \gamma_- \\ &\quad + \theta^+ \theta^- f\end{aligned}$$

where we also expand  $y^\pm = x^\pm - i\theta^\pm \bar{\theta}^\pm$  around  $x^\pm$ .

Also, take the complex conjugation of  $\Phi$ , then

$$\begin{aligned}\bar{\Phi} &= \bar{\phi} + i\theta^+ \bar{\theta}^+ \partial_+ \bar{\phi} + i\theta^- \bar{\theta}^- \partial_- \bar{\phi} - \theta^+ \theta^- \bar{\theta}^- \bar{\theta}^+ \partial_+ \partial_- \bar{\phi} \\ &\quad - \bar{\theta}^+ \bar{\gamma}_+ - i\bar{\theta}^+ \theta^- \bar{\theta}^- \partial_- \bar{\gamma}_+ - \bar{\theta}^- \bar{\gamma}_- - i\bar{\theta}^- \theta^+ \bar{\theta}^+ \partial_+ \bar{\gamma}_- \\ &\quad + \bar{\theta}^- \bar{\theta}^+ f\end{aligned}$$

Define the D-term as

$$S_D = \int d^2x d^4\theta \bar{\Phi} \Phi, \text{ where } d^4\theta = d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+$$

$\int d^4\theta \Rightarrow$  pick up the coefficient of  $\theta^4 = \theta^+ \theta^- \bar{\theta}^- \bar{\theta}^+$  in  $\bar{\Phi} \Phi$ .

$$\bar{\Psi} \Psi |_{\theta^4} = -\bar{\phi} \partial_+ \partial_- \phi + \partial_+ \bar{\phi} \partial_- \phi + 2 \bar{\phi} \partial_+ \phi - \partial_+ \bar{\phi} \phi + i \bar{\psi}_+ \partial_- \gamma_+ - i \partial_+ \bar{\psi}_+ \gamma_+ + i \bar{\psi}_- \partial_+ \gamma_- - i \partial_+ \bar{\psi}_- \gamma_- + |f|^2 \quad (7)$$

$$+ i \bar{\psi}_+ \partial_- \gamma_+ - i \partial_+ \bar{\psi}_+ \gamma_+ + i \bar{\psi}_- \partial_+ \gamma_- - i \partial_+ \bar{\psi}_- \gamma_- + |f|^2$$

By partial integral,

$$S_D = \int d^2x \left( |\partial_0 \phi|^2 - |\partial_1 \phi|^2 + i \bar{\psi}_- (\partial_0 + \partial_1) \gamma_- + i \bar{\psi}_+ (\partial_0 - \partial_1) \gamma_+ + |f|^2 \right)$$

which is the kinetic term for the complex scalar field  $\phi$  and the Dirac fields  $\gamma_\pm$ ,  $\bar{\psi}_\pm$ , and  $f$  is an auxiliary field.

Define the F-term as

$$S_F = \int d^2x d^2\theta W(\bar{\Psi}) + \text{complex conjugation term}$$

$\int d^2\theta \Rightarrow$  pick up the coefficient of  $\theta^2 = \theta^+ \theta^-$  in  $W(\bar{\Psi})$

$$W(\bar{\Psi})|_{\theta^2} = W'(\phi) f - W''(\phi) \bar{\psi}_+ \gamma_-$$

$$W(\bar{\Psi})|_{\theta^2} \sim W'(\phi) \theta^+ \theta^- f|_{\theta^2} + \frac{1}{2} W''(\phi) (\theta^+ \bar{\psi}_+ + \theta^- \bar{\psi}_-)^2|_{\theta^2}$$

Then the F-term is ⑧

$$S_F = \int d^2x (W'(\phi)f - W''(\phi)\gamma_+ \gamma_- + \bar{W}'(\bar{\phi})\bar{f} - \bar{W}''(\bar{\phi})\bar{\gamma}_- \bar{\gamma}_+)$$

The action of chiral theory is

$$\begin{aligned} S &= S_D + S_F \\ &= \int d^2x \left( |D_0\phi|^2 - |D_1\phi|^2 - |W'(\phi)|^2 + i\bar{\gamma}_-(\gamma_0 + \gamma_1)\gamma_- \right. \\ &\quad + i\bar{\gamma}_+(\gamma_0 - \gamma_1)\gamma_+ - W''(\phi)\gamma_+ \gamma_- \\ &\quad \left. - \bar{W}''(\bar{\phi})\bar{\gamma}_- \bar{\gamma}_+ + |f + \bar{W}'(\bar{\phi})|^2 \right) \end{aligned}$$

the term  $|f + \bar{W}'(\bar{\phi})|^2$  can be eliminated by the equation of motion for  $f$ , which is  $f = -\bar{W}'(\bar{\phi})$ .

where  $|W'(\phi)|^2$  is potential for  $\phi$  and  $W''(\phi)$  is the Yukawa interaction for fermion mass term  $W''(\phi)\gamma_+ \gamma_-$ .

The action  $S$  is invariant under the susy transform

$$\delta = E_+ Q - E_- Q + \bar{E}_+ \bar{Q} - \bar{E}_- \bar{Q}$$

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$$\bar{D}_{\pm} S \bar{\Phi} = S \bar{D}_{\pm} \bar{\Phi} = 0, S \bar{\Phi} \text{ is chiral}$$

The variations of fields are

$$\begin{cases} \delta \phi = \epsilon_+ \gamma_- - \epsilon_- \gamma_+ \\ \delta \psi_{\pm} = \pm 2i \bar{E}_{\mp} \partial^{\pm} \phi + \epsilon_{\pm} f \\ \delta f = -2i \bar{E}_+ \partial_- \gamma_+ - 2i \bar{E}_- \partial_+ \gamma_- \end{cases}$$

For different parameters  $\epsilon_1$  and  $\epsilon_2$ ,

$$[\delta_1, \delta_2] = 2i(\epsilon_- \bar{E}_2 - \epsilon_2 \bar{E}_-) \partial_+ + 2i(\epsilon_{1+} \bar{E}_{2+} - \epsilon_{2+} \bar{E}_{1+}) \partial_-$$

We say the action  $S$  has  $N=(2,2)$  supersymmetry.

The conserved currents are

$$G_{\pm}^0 = 2 \partial_{\pm} \bar{\Phi} \psi_{\pm} \mp i \bar{\psi}_{\mp} \bar{W}'(\bar{\Phi})$$

$$G_{\pm}^1 = \mp 2 \partial_{\pm} \bar{\Phi} \psi_{\pm} - i \bar{\psi}_{\mp} \bar{W}'(\bar{\Phi})$$

$$\bar{G}_{\pm}^0 = 2 \bar{\psi}_{\pm} \partial_{\pm} \phi \pm i \gamma_{\mp} W'(\phi)$$

$$\bar{G}_{\pm}^1 = \mp 2 \bar{\psi}_{\pm} \partial_{\pm} \phi \pm i \gamma_{\mp} W'(\phi)$$

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The conserved charges are

$$Q_{\pm} = \int dx^i G_{\pm}^0, \quad \bar{Q}_{\pm} = \int dx^i \bar{G}_{\pm}^0$$

The supercharges transform as spinors

$$Q_{\pm} \mapsto e^{\mp \frac{1}{2}\gamma} Q_{\pm}, \quad \bar{Q}_{\pm} \mapsto e^{\mp \frac{1}{2}\gamma} \bar{Q}_{\pm}$$

Introduce <sup>vector</sup> $\sqrt{R}$ -rotations and axial  $R$ -rotations for superfield

$$e^{i\alpha F_V}: \bar{\Phi}(x^u, \theta^{\pm}, \bar{\theta}^{\pm}) \mapsto e^{i\alpha q_V} \bar{\Phi}(x^u, e^{-i\alpha} \theta^{\pm}, e^{i\alpha} \bar{\theta}^{\pm})$$

$$e^{i\beta F_A}: \bar{\Phi}(x^u, \theta^{\pm}, \bar{\theta}^{\pm}) \mapsto e^{i\beta q_A} \bar{\Phi}(x^u, e^{\mp i\beta} \theta^{\pm}, e^{\pm i\beta} \bar{\theta}^{\pm})$$

where the numbers  $q_V, q_A$  are vector  $R$ -charge and axial  $R$ -charge of  $\bar{\Phi}$ .

By setting  $\bar{\Phi}$  to 0 R-charge, the action has global axial  $R$ -symmetry under

$$\bar{\Phi}(x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm}) \mapsto \bar{\Phi}(x^{\pm}, e^{\mp i\alpha} \theta^{\pm}, e^{\pm i\alpha} \bar{\theta}^{\pm})$$

The axial R-symmetry can be realized as the transformation ⑪ of the component fields,

$$\phi \mapsto \phi, \psi_{\pm} \mapsto e^{\mp i\alpha} \psi_{\pm}$$

The corresponding current is

$$J_A^0 = \bar{\psi}_+ \psi_+ - \bar{\psi}_- \psi_-$$

$$J_A^1 = -\bar{\psi}_+ \psi_+ - \bar{\psi}_- \psi_-$$

The conserved charge is  $F_A = \int dx' J_A^0$ , and the axial R-rotations rotates the supercharges as

$$Q_{\pm} \mapsto e^{\mp i\alpha} Q_{\pm}, \bar{Q}_{\pm} \mapsto e^{\pm i\alpha} \bar{Q}_{\pm}$$

Vector R-rotation: since  $\Theta^2$  has vector R-charge -2 as  $\Theta^2 \mapsto \Theta^2 e^{-2i\alpha}$ , so the F-term is invariant under vector R-rotation if and only if  $W(\tilde{\Phi})$  has vector R-charge +2. Then assume

$$W(\tilde{\Phi}) \sim \tilde{\Phi}^n$$

it requires that  $\tilde{\Phi}$  has vector R-charge  $\frac{2}{n}$ .

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Then the vector R-rotation is

$$\phi \mapsto e^{\frac{2}{n}i\alpha}\phi, \gamma_{\pm} \mapsto e^{(\frac{2}{n}-1)i\alpha}\gamma_{\pm}$$

The conserved currents are

$$J_V^0 = \frac{2i}{n} ((\partial_0 \bar{\phi})\phi - \bar{\phi} \partial_0 \phi) - (\frac{2}{n}-1)(\bar{\gamma}_+ \gamma_+ + \bar{\gamma}_- \gamma_-)$$

$$J_V^1 = \frac{2i}{n} (-(\partial_1 \bar{\phi})\phi + \bar{\phi} \partial_1 \phi) + (\frac{2}{n}-1)(\bar{\gamma}_+ \gamma_+ - \bar{\gamma}_- \gamma_-)$$

The conserved charge is  $F_V = \int d\mathbf{x} J_V^0$

The supercharges transform as

$$Q_{\pm} \mapsto e^{-i\alpha} Q_{\pm}, \bar{Q}_{\pm} \mapsto e^{i\alpha} \bar{Q}_{\pm}.$$

The axial and vector R-rotations commute.

## 5. Twisted Chiral Superfield Theory

The action  $S_{\text{twisted}}$  is defined as

$$S_{\text{twisted}} = - \int d^2x d\theta^4 \bar{\Phi}^t \bar{\Phi}^t + \left( \int d^2x d^2\bar{\theta} \tilde{W}(\bar{\Phi}^t) + \text{complex conjugation term} \right)$$

The chiral and twisted chiral superfields are related by (13)

$$\theta^- \longleftrightarrow -\bar{\theta}^-$$

$$\text{So } d\theta^- d\bar{\theta}^- = - d\bar{\theta}^- d\theta^-$$

Make the replacements by

$$\phi \mapsto \phi^t, \gamma_+ \mapsto \bar{\chi}_+, \gamma_- \mapsto -\bar{\chi}_-, f \mapsto -e$$

$$E_+ \mapsto -\bar{E}_+, Q_- \mapsto \bar{Q}_-, F_V \mapsto F_A \text{ and } F_A \mapsto F_V.$$

The susy transformations are

$$\delta \phi^t = \bar{E}_+ \chi_- - E_- \bar{\chi}_+$$

$$\delta \bar{\chi}_+ = 2i\bar{E}_- \partial_+ \phi^t + \bar{E}_+ e$$

$$\delta \chi_- = -2iE_+ \partial_- \phi^t + E_- e$$

$$\delta e = -2iE_+ \partial_- \bar{\chi}_+ - 2i\bar{E}_- \partial_+ \chi_-$$

The conserved charges are

$$Q_+ = \int dx^i (2\partial_+\bar{\phi}^t \bar{\chi}^+ + i\bar{\chi}_- \bar{W}'(\bar{\phi}^t))$$

$$\bar{Q}_+ = \int dx^i (2\chi_+ \partial_+ \phi^t - i\chi_- \tilde{W}'(\phi^t))$$

$$Q_- = \int dx^i (-2\bar{\chi}_- \partial_- \phi^t - i\bar{\chi}_+ \tilde{W}'(\phi^t))$$

$$\bar{Q}_- = \int dx^i (-2\partial_- \bar{\phi}^t) \chi_- + i\chi_+ \bar{W}'(\bar{\phi}^t))$$

The vector and axial R-symmetry generators are

$$F_V = \int dx^i (-\bar{\chi}_+ \chi_+ - \bar{\chi}_- \chi_-)$$

$$F_A = \int dx^i \left( \frac{2i}{n} ((\partial_0 \bar{\phi}^t) \phi^t - \bar{\phi}^t \partial_0 \phi^t) - \left( \frac{2}{n} - 1 \right) (-\bar{\chi}_+ \chi_+ + \bar{\chi}_- \chi_-) \right)$$

## 6. Physical expression of Mirror Symmetry

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For  $N=(2,2)$  field theories, we have four generators of supersymmetry  $Q_+$ ,  $Q_-$ ,  $\bar{Q}_+$ ,  $\bar{Q}_-$ , and  $H$  for time translations  $\frac{\partial}{\partial t^0}$ ,  $P$  for spatial translation  $\frac{\partial}{\partial x^i}$ ,  $M$  for Lorentz rotation  $x^0 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^0}$ . And if the action has vector and axial R-symmetry, we also need  $F_V$  and  $F_A$ .

Define  $SO = [\hat{\delta}, 0]$

with  $\hat{\delta} = i(E_+ Q_- - E_- Q_+ - \bar{E}_+ \bar{Q}_- + \bar{E}_- \bar{Q}_+)$

$\hat{\delta}^\dagger = -\hat{\delta}$ ,  $\bar{Q}_\pm^\dagger = Q_\pm$ ,  $(SO)^\dagger = SO^*$ .

The algebra is given by

$$Q_+^2 = Q_-^2 = \bar{Q}_+^2 = \bar{Q}_-^2 = 0, \{Q_\pm, \bar{Q}_\pm\} = H \pm P$$

$$\{\bar{Q}_+, \bar{Q}_-\} = \{Q_+, Q_-\} = 0, \{Q_-, \bar{Q}_+\} = \{Q_+, \bar{Q}_-\} = 0$$

$$[iM, Q_\pm] = \mp Q_\pm, [iM, \bar{Q}_\pm] = \mp \bar{Q}_\pm$$

$$[iF_V, Q_\pm] = -iQ_\pm, [iF_V, \bar{Q}_\pm] = i\bar{Q}_\pm$$

$$[iF_A, Q_\pm] = \mp iQ_\pm, [iF_A, \bar{Q}_\pm] = \pm i\bar{Q}_\pm$$

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If  $\{\bar{Q}_+, \bar{Q}_-\} = Z$ ,  $\{Q_+, Q_-\} = \bar{Z}^*$

$\{Q_-, \bar{Q}_+\} = \tilde{Z}$ ,  $\{Q_-, \bar{Q}_-\} = \tilde{Z}^*$

then central charge  $Z$  must be zero if  $F_V$  is conserved

$\tilde{Z} = 0$  if  $F_A$  is conserved.

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The  $N=(2,2)$  supersymmetry algebra is invariant under

$$Q_- \longleftrightarrow \bar{Q}_-$$

$$F_V \longleftrightarrow F_A$$

$$Z \longleftrightarrow \tilde{Z}$$

Two  $N=(2,2)$  susy field theories are mirror if they exchange the generators of algebra according to the above equation.

For mirror theories, a chiral multiplet of one theory is mapped to a twisted chiral multiplet of the mirror theory.