EE: 1907.08126v1

9/26/2019

Some extra notes on ~~[illegible]~~ reduced entropy

History: 1970-Bekenstein-Hawking black hole entropy

Now understood that in vacuum of QFT even in flat space a region of space is mixed state.

Rindler space in Gibbs state WRT boost gen.

1980's-90's - 't Hooft, Bombelli Koul-Lee-Sorkin Susskind Srednicki

leading term in entropy of region is propotional to area. (UV divergent)

Hence entanglement of quatum fields across the horizon partly explains BH entropy

90's much work is done on EE in field Theory

2000's EE became a standard tool for CM ~~a~~ theorists to characterize phases of many body systems.

ex. Calabrese-Cardy EE in 2d CFTS

Hastings et al. gapped systems satisfy an area law.

Kitaev-Preskill/Levin-Wen's topological EE

Li-Haldanes use of entanglement spectrum to characterize fractional Q Hall states.

in this decade Ryu-Takayanagi conjectured a simple formula for EE in Holographic Theories.

Shannon entropy

Consider a classical Theory with a discrete state space.

our knowledge of the system, in particular of the state, is described by a probability distribution $\vec{p}$ with,

$$p_a \geq 0 \qquad \sum_a p_a = 1$$

The expectation value of an observable is

$$\langle O_a \rangle_{\vec{p}} = \sum_a O_a p_a$$

The shannon entropy is

$$S(p_a) \equiv - \sum_a p_a \ln p_a$$

Notice the connection to the boltzmann entropy supposing all states are equally likely Then Given the number of states Ω

$$p_a = 1/\Omega \Rightarrow -\sum_{a=1}^{\Omega} \frac{1}{\Omega} \ln \frac{1}{\Omega} = -\sum_{a=1}^{\Omega} \frac{1}{\Omega}(\ln 1 - \ln \Omega)$$

$$\boxed{\frac{S}{k} = \ln \Omega}$$

The shannon entropy detects uncertainty in the state:

$$S(p_a) = 0 \Leftrightarrow p_a = \delta_{a a_0} \text{ for some state } a_0$$

This is true if and only if all observables have vanishing variance $\Delta O = \langle O^2 \rangle - \langle O \rangle^2 = 0$

otherwise $S(p) > 0$

$$\langle O_a\rangle^2 - \langle O_a^2\rangle$$

$$= \sum_{a,b} O_a P_a O_b P_b - \sum_a O_a^2 P_a \qquad \text{Put } P_a = \delta_{aa_0}$$

$$= O_{a_0}^2 - O_{a_0}^2 = 0$$

and $S = -\sum P_a \ln P_a = -\sum \delta_{aa_0} \ln \delta_{aa_0}$

$$= \begin{cases} 0 & \text{if } a \neq a_0 \\ -\delta_{a_0a_0} \ln \delta_{aa_0} \\ = 1 \ln 1 = 0 & \text{if } a_0 = a \end{cases}$$

$\Rightarrow$ $S = 0$ for $P_a = \delta_{aa_0}$

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note 2 Things.

1.) Shannon entropy is extensive
i.e. if A & B are independent
i.e. The joint distribution satisfies
$$P_{AB} = P_A \otimes P_B \quad \rightarrow \quad (P_{AB})_{ab} = (P_A)_a (P_B)_b$$
Then The entropies Add

$$S(P_{AB}) = -\sum_{ab} P_{AB\,ab} \ln P_{AB\,ab}$$

$$= -\sum_{ab} P_a P_b \ln P_a + P_a P_b \ln P_b \qquad \text{use } \sum_{P_a} P_a = 1, \quad \sum_b P_b = 1$$

$$= -\left(\sum_a (\ln P_a) P_a + \sum_b P_b \ln P_b\right)$$

$$S(P_{AB}) = +S(P_A) + S(P_B) \quad \checkmark$$

note if we have N independent copies of A with identical distributions Then $S(P_{TOT}) = N S(P_A)$

2.) The Shannon noiseless coding Theorem States The state of our system can be specified using a binary code requiring on avg. $S(P_a)/\ln 2$ bits.

"All forms of ~~entropy~~ information $\longrightarrow$

can be interconverted as long as the entropies match"

This is very useful in the sense that we do not need to create the coding to know how many bits we need.

Joint distributions:

Consider a general joint distribution P_{AB}
The ~~mage~~ marginal distribution is defined by integrating out or tracing over 1 of the sub systems

$$(P_A)_a = \sum_b (P_{AB})_{ab}$$

P_a gives the right ~~distribution~~ distribution for any observable which depends only on the set of states $\{a\}$ $\quad O_{ab} = O_a$

$$\langle O_{ab} \rangle_{P_{ab}} = \langle O_a \rangle_{P_a}$$

$$\sum_{ab} O_{ab} P_{ab} = \sum_{ab} O_a P_{ab}$$

$$\sum_a \left(O_a \sum_b P_{ab} \right) = \sum O_a P_a = \langle O_a \rangle_{P_a} \checkmark$$

For a state b of B with $P_b \neq 0$
The conditional probability on A $P_{A|b}$ is

$$(P_{A|b})_a = \frac{(P_{AB})_{ab}}{(P_B)_b}$$

its entropy avg. over b is

$$\langle S(P_{A|b})\rangle_{P_b} = \sum_b P_b\, S(P_{A|b})$$

$$= \sum_b P_b(-P_{A|b} \ln P_{A|b})$$

$$= -\sum_{a,b} P_b\left(\frac{P_{AB\,ab}}{P_{B\,b}} \ln \frac{P_{AB\,ab}}{P_{B\,b}}\right)$$

$$= -\sum_{a,b} P_{AB\,ab} \ln P_{AB} - P_{AB\,ab} \ln P_{B\,b}$$

$$= S(P_{AB}) - S(P_B)$$

its easier from this point on to write $S(B) \equiv S(P_B)$. This quantity is called the conditional entropy. This is the amount, on average, of entropy which remains to be known about state A ~~are~~ after knowing state B.

note that since $H(A|B) \equiv \langle S(P_{A|b})\rangle_{P_b} \geq 0$ so is $S(AB) - S(B) \geq 0$ and if $S(AB) = 0$ then so is $S(B)$. This fails in the quantum setting.

A simple example to better understand this is a phone call between alice and bob when the reception is bad. if the message is A, a set of letters and the recieved message is B another set of letters then our machinery helps us to understand the amount of information gained during the call.

The marginal distribution P_B is the probability distribution ~~of~~ describing the probability bob heard B when Alice said A.

$$P_B = \sum_a P_{AB\,ab}$$

Bobs estimate of the probability that Alice said A after hearing B is the conditional probability

$$P_{A|B} = \frac{P_{AB\,ab}}{P_{B\,b}}$$

The Shannon entropy gives, from bobs point of view ~~the~~ an estimate of the remaining entropy in Alices signal

$$S_{X|Y} = -\sum_{ab} P_{AB} \ln P_{AB}.$$

and as described on the previous page the average over B of $S_{X|Y}$ is the avg. remaining entropy in Alices message.

Since S_A is the total information content about state A, (or Alices message);

$S_{AB} - S_B$ is the ~~information~~ information bob still does not know about the message ~~or~~ or state A then the remaining information ~~is~~ which bob does gain after hearing/observing B is

$$I(A|B) = S_A - S_{AB} + S_B$$

~~A~~ This is called the mutual information. it tells us how much we learn about A by measuring B.

more motivation:

- Why Study this Problem?

- intrinsic entropy of a black hole is $S_{BH} = \frac{1}{4} M_{pl}^2 A$
M_{pl} is The Planck mass A is The surface area.

- At This time people were still wondering if S_{BH} has anything to do with The # of quantum states accesible to The black hole.

- As a black hole shrinks it emits hawking radiation whose entropy S_{HR} is $S_{HR} = \# S_{BH}$ Where # is $O(1)$.
- Calculating S_{HR} is done via counting quantum states

- obtaining $S \sim A$ shows getting The amount of missing information represented by S_{BH} as an answer is what we would expect in a flat space if we did not permit ourselves access to The interior of a Sphere.

Entropy & Area

Casey Cartwright

- Could talk about a lot of different topics
  Replica trick, CFT's with holographic duals etc.
- Will be more instructive to focus on a single result.
- I constantly read references to early works on entropy in QFT. Decided to discuss
  Srednicki 9303048.

Goal: Show that ground state density matrix for a free massless free field traced over d.o.f residing in a sphere: resulting entropy is proportional to area.

- Free massless scalar QF.
  represent acoustic modes of a crystal
  any 3D system with $\omega = c|\vec{k}|$
- in non degenerate vacuum state
- form ground state density matrix $\rho_0 = |0\rangle\langle 0|$
- tr over d.o.f inside sphere of radius R S^2
- ρ_{out} resulting density matrix depends on d.o.f outside S^2

$S = -\rho_{out} \operatorname{tr} \rho_{out}$ $\quad S(R) \propto R^3 ? \; R^2 ?$

Entropy is ~~extens~~ extensive i.e. depends on system size.

i.e. we expect $S \propto R^3$

We will see that in fact $S = \kappa M^2 A$

with A - area M - UV cutoff κ - dim numerical constant.

lets start with a simple ex.

two coupled HO (harmonic oscillators)

$$H = \frac{1}{2}\left(P_1^2 + P_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2 \right]$$

Ground State Wave function:

$$\psi_0(x_1,x_2) = \frac{(\omega_+\omega_-)^{1/4}}{\pi^{1/2}} e^{-\frac{(\omega_+ x_+^2 + \omega_- x_-^2)}{2}}$$

$$x_\pm = \frac{(x_1 \pm x_2)}{\sqrt{2}} \qquad \omega_+ = k_0^{1/2} \qquad \omega_- = (k_0 + 2k_1)^{1/2}$$

transformation matrix $\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_\pm}$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \partial_+ \\ \partial_- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \partial_+ + \partial_- \\ \partial_+ - \partial_- \end{pmatrix}$$

$$P_1^2 = (-i\hbar\partial_1)^2 \quad \hbar \to 1 \quad P_2^2 = (-i\hbar\partial_2)^2$$

$$P_1^2 = -\partial_1^2 = -\frac{1}{2}(\partial_+ + \partial_-)^2 = -\frac{1}{2}(\partial_+^2 + 2\partial_+\partial_- + \partial_-^2)$$

$$P_2^2 = -\frac{1}{2}(\partial_+^2 - 2\partial_+\partial_- + \partial_-^2)$$

$$H = \frac{1}{2}\left[-\partial_+^2 - \partial_-^2 + k_0(x_+^2 + x_-^2) + 2k_1 x_-^2\right]$$

I will leave proof this is the ground state to you!

Now we can form the density matrix $\rho = \psi\psi^*$ and trace out the "inside" oscillator leaving the "outside" oscillator

$$\rho_{out} = \int_{-\infty}^{\infty} dx_1\, \psi(x_1,x_2)\psi^*(x_1,x_2')$$

$$= \int_{-\infty}^{\infty} dx_1 \frac{(\omega_+\omega_-)^{1/2}}{\pi} e^{-\left(\frac{\omega_+}{2}(x_1^2+2x_1x_2+x_2^2)+\frac{\omega_-}{2}(x_1^2-2x_1x_2+x_2^2)\right)\frac{1}{2}}$$
$$e^{-\frac{1}{2}\left(\frac{\omega_+}{2}(x_1^2+2x_1x_2'+x_2'^2)+\frac{\omega_-}{2}(x_1^2-2x_1x_2'+x_2'^2)\right)}$$

$$= \frac{(\omega_+\omega_-)^{1/2}}{\pi} e^{-\frac{1}{4}(\omega_+x_2^2+\omega_-x_2^2+\omega_+x_2'^2+\omega_-x_2'^2)}$$

$$\times \underbrace{\int_{-\infty}^{\infty} dx_1\, e^{-\frac{\omega_+}{4}(2x_1^2+\cancel{4}\,2x_1(x_2+x_2'))-\frac{\omega_-}{4}(2x_1^2-\cancel{4}\,2x_1(x_2+x_2'))}}$$

$$\int_{-\infty}^{\infty} dx_1\, e^{-x_1^2\underbrace{\left(\frac{\omega_+}{2}+\frac{\omega_-}{2}\right)}_{\alpha}-x_1\underbrace{(x_2+x_2')(\omega_+-\omega_-)/2}_{2\beta}}$$

$$e^{-\alpha(x_1^2+2\frac{\beta}{\alpha}x_1+\beta^2/\alpha^2)+\beta^2/\alpha}$$

$$e^{\beta^2/\alpha^3}\; e^{-\alpha(x_1+\beta/\alpha)^2}$$

$x_1 \to x_1 - \beta/\alpha$

$x_1 \to (\sqrt{\alpha})^{-1}x_1'$

$$(\sqrt{\alpha})^{-1}\int_{-\infty}^{\infty} e^{-x_1^2} = (\sqrt{\alpha})^{-1}\sqrt{\pi}$$

$$= \frac{(2\omega_+\omega_-)^{1/2}}{(\omega_++\omega_-)^{1/2}}\pi^{-1/2} e^{-\frac{1}{4}((\omega_++\omega_-)x_2^2+(\omega_++\omega_-)x_2'^2)}\; e^{\frac{2(x_2+x_2')^2(\omega_+-\omega_-)^2}{16\,\omega_+\omega_-}\frac{1}{\omega_++\omega_-}}$$

$$e^{\frac{\beta}{2}(x_2^2+2x_2x_2'+x_2'^2)}$$

$$\beta = \frac{1}{4}\frac{(\omega_+-\omega_-)^2}{\omega_++\omega_-}$$

$\Rightarrow$ look at $e^{\cdots}$

$$-\frac{1}{4}(\omega_+ + \omega_-)x_2^2 + \frac{\beta}{2}x_2^2 \quad + x_2 \to x_2' \quad + \beta x_2 x_2'$$

$$= -\left(\tfrac{1}{2}(\omega_+ + \omega_-) - \beta\right) x_2^2/2$$

$$= \left[\frac{\frac{1}{2}(\omega_+ + \omega_-)^2 - \frac{1}{4}(\omega_+ - \omega_-)^2}{\omega_+ + \omega_-}\right] x_2^2/2 \quad + \cdots$$

$$= \tfrac{1}{2}\omega_+^2 + \omega_+\omega_- + \frac{\omega_-^2}{2} - \tfrac{1}{4}\omega_+^2 + \tfrac{1}{2}\omega_+\omega_- - \tfrac{1}{4}\omega_-^2$$

$$= \omega_+\omega_- + \tfrac{1}{2}\omega_+\omega_- + \tfrac{1}{4}\omega_+^2 + \tfrac{1}{4}\omega_-^2 \quad + \omega_+\omega_- - \omega_+\omega_-$$

$$= 2\omega_+\omega_- - \tfrac{1}{2}\omega_+\omega_- + \tfrac{1}{4}\omega_+^2 + \tfrac{1}{4}\omega_-^2$$

all together

$$= -\left[\underbrace{\frac{2\omega_+\omega_-}{\omega_+ + \omega_-} + \frac{1}{4}\frac{(\omega_+ - \omega_-)^2}{\omega_+ + \omega_-}}\right] x_2^2/2 + x_2 \to x_2' + \beta x_2 x_2'$$

$$= -\gamma(x_2^2 + x_2'^2) + \beta x_2 x_2'$$

$$\overset{?}{=} \quad \boxed{\gamma - \beta = \frac{2\omega_+\omega_-}{\omega_+ + \omega_-}}$$

I'll leave this to you to show.

$\Rightarrow$ The full result is

$$\rho_{out} = \pi^{-1/2}(\gamma - \beta)^{1/2} e^{-\gamma(x_2^2 + x_2'^2) + \beta x_2 x_2'}$$

Now we would like to find the eigenvalues of ρ_{out} (P_n)

$\longrightarrow$

This can be written as an integral eqn.

$$\int_{-\infty}^{\infty} dx_2' \, \rho_{out}(x_2, x_2') f_n(x_2') = P_n f_n(x_2)$$

i.e. Think $\sum_j A_{ij} f_{nj} = P_n f_{ni}$ here we have a continuous label $\therefore$ integration not summation. if we find them the Von-neuman entropy is $S = -\sum P_n \log P_n$

Consider $f = 1$

$$\int_{-\infty}^{\infty} dx_2' \, \pi^{-1/2}(\gamma-\beta)^{1/2} e^{-\gamma(x_2^2 + x_2'^2) + \beta x_2 x_2'}$$

$$= -\gamma x_2'^2 + \beta x_2 x_2' - \gamma x_2^2$$

$$-\gamma(x_2'^2 - \underbrace{\beta/\gamma}_{2\alpha} x_2 x_2' + x_2^2).$$

$$-\gamma((x_2' + \alpha)^2 - \alpha^2 + x_2^2)$$

$$\to -\gamma\alpha^2 + \gamma x_2^2$$

$$-\frac{\beta^2}{4\gamma} x_2^2 + \gamma x_2^2$$

This can be found by guessing (Srednicki is a good guesser!) i will ~~way~~ verify the first one and leave the general Proof to you. (one way would be induction)

or you could just do it directly.

Consider $f_0 = e^{-\alpha x^2/2}$, $\alpha = (\gamma^2 - \beta^2)^{1/2}$

$$\Rightarrow \int_{-\infty}^{\infty} dx' \, \pi^{-1/2} (\gamma - \beta)^{+1/2} e^{-\frac{\gamma}{2}(x^2 + x'^2) + \beta x x' - \frac{\alpha x'^2}{2}}$$

complete the square

$$-x'^2\left(\frac{\gamma}{2} + \alpha/2\right) + \beta x x' + -\frac{\gamma}{2} x^2$$

$$-\left(\frac{\gamma}{2} + \alpha/2\right)\left(x'^2 + -\frac{\beta x x'}{(\frac{\gamma}{2} + \alpha/2)} + \frac{\gamma/2\, x^2}{(\frac{\gamma}{2} + \alpha/2)}\right)$$

$$-\frac{\beta x}{(\frac{\gamma}{2} + \alpha/2)} \frac{1}{2} = b$$

$$x'^2 + 2bx' + \frac{\gamma x^2/2}{(\gamma + \alpha/2)}$$

$$-\left(\frac{\gamma}{2} + \alpha/2\right)\left((x' + b)^2 - b^2 + \frac{\gamma x^2}{2(\gamma + \alpha/2)}\right)$$

$$x' \to x' - b \qquad + \frac{\beta^2 x^2}{4(\frac{\gamma}{2} + \alpha/2)} + -\frac{\gamma x^2}{2(2\frac{\gamma}{2} + \alpha/2)}$$

$$x' = \frac{y}{\sqrt{\frac{\gamma}{2} + \alpha/2}}$$

$$\pi^{-1/2} (\gamma - \beta)^{+1/2} \left(\frac{\gamma}{2} + \alpha/2\right)^{-1/2} \underbrace{\int_{-\infty}^{\infty} dy \, e^{-y^2}}_{\sqrt{\pi}} \; e^{+\frac{\frac{\beta^2 x^2}{4} - \frac{\gamma}{2} x^2 (\frac{\gamma}{2} + \alpha/2)}{\frac{\gamma}{2} + \alpha/2}}$$

$$= \left(\frac{\gamma - \beta}{\gamma + \alpha}\right)^{1/2} \sqrt{2}\, e^{-\frac{1}{2}\alpha x^2} \qquad (\gamma - \beta)^{+1/2} \left(\frac{\gamma}{2} + \alpha/2\right)^{-1/2} e^{+\frac{\frac{\beta^2 x^2}{4} - \frac{\gamma}{2} x^2 (\frac{\gamma}{2} + \alpha/2)}{\frac{\gamma}{2} + \alpha/2}}$$

$$\gamma + \alpha = \underbrace{\omega_-^2 + 6\omega_-\omega_+ + \omega_+^2 + 4\omega_-\sqrt{\omega_-\omega_+} + 4\omega_+\sqrt{\omega_+\omega_-}}_{\times \frac{1}{4\omega_+ + \omega_-}} \qquad e^{-\frac{1}{4} \cdot 2 \frac{(-\beta^2 + \gamma^2 - \alpha\gamma)}{\gamma + \alpha} x^2}$$

$$= \frac{(2\sqrt{\omega_-\omega_+} + (\omega_- + \omega_+))^2}{4(\omega_+ + \omega_-)} = \frac{(\sqrt{\omega_-} + \sqrt{\omega_+})^4}{4(\omega_+ + \omega_-)} \qquad e^{-\frac{1}{2}\left(\frac{\alpha(\alpha + \gamma)}{\gamma + \alpha}\right) x^2}$$

$$\gamma - \beta = \frac{2\omega_-\omega_+}{\omega_+ + \omega_-} \Rightarrow \left(\frac{\sqrt{2}\sqrt{\omega_-\omega_+}}{\sqrt{\omega_+ + \omega_-}} \frac{4(\omega_+ + \omega_-)}{(\sqrt{\omega_-} + \sqrt{\omega_+})^4}\right)^{+1/2} \qquad e^{-\frac{1}{2}\alpha x^2}$$

$$= \frac{4\sqrt{\omega_-\omega_+}}{(\sqrt{\omega_-} + \sqrt{\omega_+})^2}$$ which can be rewritten as

$$1 - \frac{\beta}{\gamma+\alpha} = 1 - \xi$$

altogether $\Rightarrow \rho = (1-\xi) e^{-\frac{\alpha}{2}x^2}$

with $P_0 = (1-\xi)$

The full form is $H_n(\alpha^{1/2} x) e^{-\frac{\alpha}{2}x^2} = f_n(x)$ eigenf$\triangle$'s

$(1-\xi)\xi^n$ eigenvalues.

This shows that ρ_{out} is ~~equiva~~ equivalent to a thermal density matrix for a single harmonic oscillator with frequency α and temp $T = \alpha / \log(1/\xi)$

$$\rightarrow \quad P_n = \left(1 - e^{-\frac{\hbar\omega}{kT}}\right) e^{-\frac{n\hbar\omega}{kT}}$$

$$S = -\sum_n P_n \log P_n = -\sum_n (1-\xi)\xi^n \log (1-\xi)\xi^n$$

$$= -\sum_n (1-\xi)\xi^n \left(\log \xi^n + \log(1-\xi)\right)$$

$$= -(1-\xi)\log\xi \sum_n n\xi^n - (1-\xi)\log(1-\xi)\sum_n \xi^n$$

$$- \log(1-\xi)$$

$$= + \frac{\xi \log \xi}{(-1+\xi)} - \log(1-\xi)$$

S is only a ratio of K_1/K_0

Now extend our analysis to N coupled harmonic oscillators.

$$H = \frac{1}{2}\sum_i^N P_i^2 + \frac{1}{2}\sum_{i,j}^N x_i x_j K_{ij}$$

K is a real symmetric matrix with positive eigenvalues.

For 2 harmonic oscillators our groundstate wave fn was

$$\psi_0 = \pi^{-1/2}(\omega_+\omega_-)^{1/4} e^{-(\omega_+ x_+^2 + \omega_- x_-^2)/2}$$

For N harmonic oscillators we can see a simple generalization $\omega_+ = K_0^{1/2}$ $\omega_- = (K_0 + 2K_1)^{1/2}$

$$\Rightarrow \quad K_{ij} = \begin{pmatrix} K_0+K_1 & -K_1 \\ -K_1 & K_0+K_1 \end{pmatrix} \Rightarrow \det K_{ij} = K_0^2 + 2K_1K_0 + K_1^2 - K_1^2$$
$$= K_0(K_0 + 2K_1)$$

$$\rightarrow (\omega_+\omega_-)^{1/4} = (\sqrt{\det K_{ij}})^{1/4}$$

if $K_{ij} = \Omega_{ik}\Omega_{kj}$ with Ω the "square root of K"

Then $(\omega_+\omega_-)^{1/4} = (\det\Omega)^{1/4}$

Since $\det\Omega = \sqrt{\det K}$

Note that if $K = U^T K_D U$ with K_D diagonal

Then $\Omega = U^T K_D^{1/2} U \Rightarrow K = U^T K_D^{1/2} U U^T K_D^{1/2} U$
$= U^T K_D U$

Note also

$$\omega_+ x_+^2 + \omega_- x_-^2 = \Omega_{ij} x_i x_j \quad \text{with } x_i = U_{ij}\tilde{x}_j, \quad \tilde{x}_j = (x_1, x_2)$$
$$x_i = (x_+, x_-)$$

$$\Rightarrow \boxed{\psi_0 = \pi^{-N/2}(\det\Omega)^{1/4} e^{-\frac{1}{2}(x_i\Omega_{ij}x_j)}}$$

Now we follow the same steps. trace out the first n "inside" oscillators

$$\rho_{out}(x_{n+1}\cdots x_N, x'_{n+1}\cdots x'_N) = \int \prod_{i=1}^{n} dx_i \, \psi_0(x_1\cdots x_n, x_{n+1}\cdots x_N) \times \psi_0^*(x_1\cdots x_n, x'_{n+1}\cdots x'_N)$$

The full Ground State wave fn is as on the back page $\psi_0 = \pi^{-N/2}(\det(\Omega))^{1/4} e^{-\frac{1}{2}(x_i \cdot \Omega_{ij} x_j)}$

to solve these integrals we can motivate ourselves with the previous solution

$$\sim e^{-\frac{\gamma x_2^2 - \gamma x_2'^2}{2} + \beta x_2 x_2'}$$

notice $\beta = \frac{1}{4}\frac{(\omega_+ - \omega_-)^2}{\omega_+ + \omega_-}$; $\gamma = \frac{2\omega_+ \omega_-}{\omega_+ + \omega_-} + \beta$

$$= \frac{2\omega_+\omega_- + \frac{1}{4}(\omega_+ - \omega_-)^2}{\omega_+ + \omega_-} = \frac{2\omega_+\omega_- + \frac{1}{4}\omega_+^2 - \frac{1}{2}\omega_+\omega_- + \frac{\omega_-^2}{4}}{\omega_+ + \omega_-}$$

when generalizing

$x_2 \to$ x. $(N-n)$ componenets

$x_2' \to$ x'. $(N-n)$ components

$\beta \to$ β $(N-n \times N-n)$ matrix

$\gamma \to$ γ $(N-n \times N-n)$ matrix

$$\beta = \frac{1}{4}\frac{(\omega_+ - \omega_-)^2}{\omega_+ + \omega_-}$$

$$\Omega = \begin{pmatrix} \omega_+ + \omega_- & \omega_+ - \omega_- \\ \omega_+ - \omega_- & \omega_+ + \omega_- \end{pmatrix}$$

$$\downarrow$$

$$\Omega = \begin{pmatrix} A\ (n\times n) & B\ (n \times N-n) \\ B^T (N-n, n) & C\ (N-n \times N-n) \end{pmatrix}$$

$= \frac{1}{4}(\omega_+ - \omega_-)(\omega_+ + \omega_-)^{-1}(\omega_+ - \omega_-)$

$= \frac{1}{4} B^T A^{-1} B$

$$\gamma = \beta + \frac{2\omega_+\omega_-}{\omega_+ + \omega_-} \quad \gamma \Rightarrow \beta - C$$

~~$\gamma - \beta = \frac{2\omega_+\omega_-}{\omega_+ + \omega_-}$~~

You can kinda see this if you look back to calculation of ρ_{out} for 2 oscillators.

We don't need to worry about the normalization since eigenvalues (ρ_{out}) $P_n \Rightarrow \sum P_n = 1$

Generalizing $\displaystyle\int_{-\infty}^{\infty} \prod_{i=1}^{N-n} \rho_{out}(x_1 \cdots x_{N-n}, x'_1 \cdots {}_{N-n}) f_n(x_1 \cdots x'_{N-n})$

$$= P_n f(x_1 \cdots x_{N-n})$$

We can see that $\det G\, \rho_{out}(G\vec{x}, G\vec{x}')$ has the same eigenvalues

let $\quad \gamma = V^T \gamma_D V \qquad V^T V = I \quad \gamma_D$ - diagonal

and put $\quad x = V^T \gamma_D^{-1/2} y$

$$\Rightarrow e^{-\frac{x\cdot\gamma\cdot x + -x'\cdot\gamma\cdot x'}{2} + x\cdot\beta\cdot x'}$$

$$\Rightarrow e^{-y\cdot\gamma_D^{-1/2}\overbrace{\underbrace{V\cdot\gamma\cdot V^T}_{\gamma_D}\gamma_D^{-1/2}}^{I} y + -y'\gamma_D^{-1/2} V\cdot\gamma\cdot V^T\gamma_D^{-1/2} y'}$$

$$+ \cancel{V^T\gamma_D^{-1/2} y}\cdot\beta\cdot V$$

$$y\cdot\gamma^{-1/2} V\cdot\beta\cdot V^T\gamma_D^{-1/2} y$$

$$= e^{-\frac{y^2 - y'^2}{2} + y\cdot\beta'\cdot y'}, \quad \beta' = \gamma_D^{-1/2} V\cdot\beta\cdot V^T\cdot\gamma_D^{-1/2}$$

Now choose $\quad y = w z \quad$ s.t. $\; w^T\beta' w = \beta'_D$

$\Rightarrow$

$$e^{-\frac{z\cdot z}{2} - \frac{z'\cdot z'}{2} + z\cdot\beta'_D\cdot z'}$$

$$= \prod_{i=n+1}^{N} e^{-(z_i^2 + z_i'^2)/2 + z_i z_i' \beta_i'}, \quad \beta_i \Rightarrow \beta'_D = \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \beta_3 & \\ & & & \ddots \end{pmatrix}$$

This is exactly as we found before with $\gamma \to 1$; $\beta \to \beta_i$

therefore the entropy is a sum over the entropy of each independent term. $\longrightarrow$

$$P_n = \prod_i (1-\xi_i)\,\xi_i^n \qquad \prod_i \equiv \prod_{i=m+1}^{N}$$

$$S = -\sum_{n=0}^{\infty} P_n \ln P_n = -\sum_n P_n \ln\left(\prod_j (1-\xi_j)\,\xi_j^n\right)$$

$$= -\sum_n P_n \left(\sum_j \ln(1-\xi_j) + \ln \xi_j^n \right)$$

$$= -\sum_n \left(\prod_i (1-\xi_i)\,\xi_i^n \sum_j \ln(1-\xi_j) + \prod_i (1-\xi_i)\,\xi_i^n \sum_j \ln \xi_j^n \right)$$

$$= -\prod_i (1-\xi_i) \sum_j \ln(1-\xi_j) \sum_n \xi_i^n + \prod_i (1-\xi_i) \sum_j \ln \xi_j \sum_n n \xi_i^n$$

$$= -\sum_j \prod_i (1-\xi_i) \frac{\ln(1-\xi_j)}{1-\xi_i} + \prod_i (1-\xi_i) \frac{\xi_i}{(1-\xi_i)^2} \sum_j \ln \xi_j$$

$$\boxed{S = -\sum_j \left(\ln(1-\xi_j) + \left(\prod_i \frac{\xi_i}{(1-\xi_i)} \right) \ln \xi_j \right)}$$

Careful I think i made a mistake in this derivation

Now we want to apply this to a scalar quantum field

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi \qquad \text{use } \eta = \mathrm{diag}(1,-1,-1,-1)$$

$$\pi = \frac{\partial \mathcal{L}}{\partial\, \partial_t\phi} = \partial_t\phi = \dot{\phi}$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \pi\dot{\phi} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$$

$$= \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}\partial_i\phi\,\partial_i\phi$$

$$\boxed{\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2}$$

$$H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int d^3x\, \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2$$

We will need to regulate the infinities so introduce partial waves

$$\varphi_{\ell m} = x\int d\Omega\, Z_{\ell m}(\theta,\phi)\phi(\vec{x})$$

$$\pi_{\ell m} = x\int d\Omega\, Z_{\ell m}(\theta,\phi)\pi(x)$$

$x = |\vec{x}|$; $Z_{\ell m}$ - real spherical harmonics

$$Z_{\ell 0} = Y_{\ell 0}\ , \quad Z_{\ell m} = \sqrt{2}\,\mathrm{Re}\,Y_{\ell m} \quad m>0$$

$$= \sqrt{2}\,\mathrm{Im}\,Y_{\ell m} \quad m<0$$

$$\int d\Omega\, Y_\ell^m Y_{\ell'}^{m'} = \delta_{\ell\ell'}\delta_{mm'}$$

We have $[\phi(x), \pi(x')] = i\delta(x - x')$ This is 3D.

$[\varphi_{\ell m}(x), \pi_{\ell' m'}(x')] =$

$= \left[x \int d\Omega \, Z_{\ell m}(\theta,\phi)\varphi(x), x' \int d\Omega' Z_{\ell' m'}(\theta',\phi')\pi(x') \right]$

$= \int d\Omega d\Omega' \left(x x' Z_{\ell m}(\theta,\phi) Z_{\ell' m'}(\theta',\phi') [\phi(x), \pi(x')] \right)$

$\left(x x' Z_{\ell m}(\theta,\phi) Z_{\ell' m'}(\theta',\phi') \, i\delta(x - x') \right)$

$= i\delta(x - x') \int \underbrace{d\Omega d\Omega' \, x x'}_{x^2 \sin\theta} Z_{\ell m}(\theta,\phi) Z_{\ell' m'}(\theta',\phi') \delta(\theta-\theta')\delta(\phi-\phi')$

$d\Omega' = \sin\theta' d\theta' d\phi'$

$= i\delta(x - x') \int d\Omega \, Z_{\ell m}(\theta,\phi) Z_{\ell' m'}(\theta',\phi)$

$= i\delta(x - x') \delta_{\ell\ell'} \delta_{mm'}$ ✓

$\phi_{\ell m} = x \int d\Omega \, Z_{\ell m}(\theta,\phi)\phi(x)$

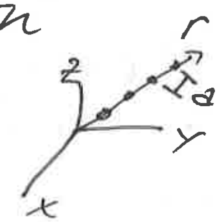
$\Rightarrow \phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \phi_{\ell m} Z_{\ell m}$

Important: as $H = \sum_{lm} H_{lm}$

With $H_{lm} = \frac{1}{2}\int_0^\infty dx \left\{ \pi_{lm}^2 + x^2\left[\frac{\partial}{\partial x}\left(\frac{\phi_{lm}(x)}{x}\right)\right]^2 + \frac{l(l+1)}{x^2}\phi_{lm}^2 \right\}$

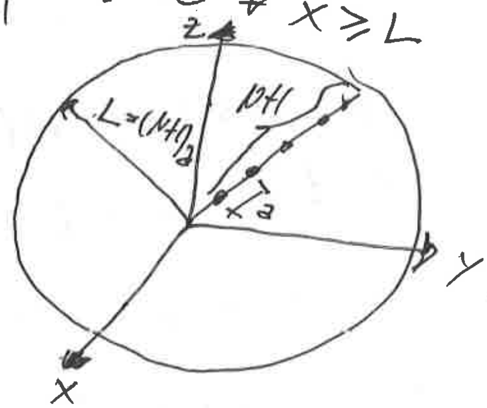
We have made no approximations or regulations.

Regulators: ultraviolet – replace radial x by a lattice of discrete points with spacing a $\Rightarrow$ UV cutoff is $M = a^{-1}$



infrared – Put the system in a spherical box of radius $L = (N+1)a$, N is a large integer; $\phi_{lm}(x) = 0 \;\forall\; x \geq L$

$\therefore$ IR cutoff $\mu = L^{-1}$



implementing this

$$H_{\ell m} = \frac{1}{2a}\sum_{j=1}^{N}\left[\pi_{\ell m,j}^2 + (j+\tfrac{1}{2})^2\left(\frac{\varphi_{\ell m,j}}{j} - \frac{\varphi_{\ell m,j+1}}{j+1}\right) + \frac{\ell(\ell+1)}{j^2}\varphi_{\ell m,j}^2\right]$$

$H_{\ell m}$ has General form of

$$H = \frac{1}{2}\sum_{i=1}^{N} P_i^2 + \frac{1}{2}\sum_{ij}^{N} x_i K_{ij} x_j$$

Can numerically compute $S_{\ell m}$ at fixed N (I tried to reproduce this but ran out of time)

Note: $S = \sum_{\ell m} S_{\ell m}(n, N)$

$H_{\ell m}$ is independent of m

$\Rightarrow$ $S_{\ell m} = S_m \quad \sum_{\ell} (2\ell+1) S_\ell(n, N)$

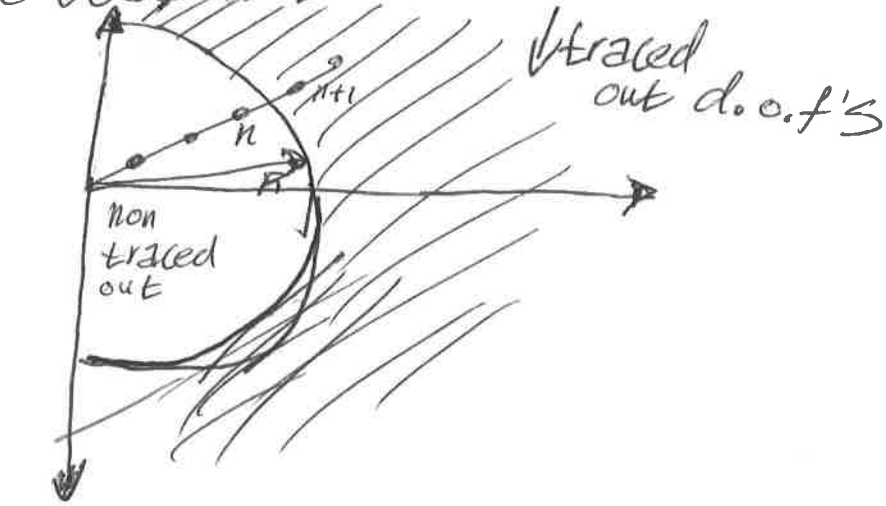
S_ℓ can be computed Perturbatively in the limit $\ell \gg N$

$S_\ell(n, N)$ is independent of N here

$$\begin{cases} S_\ell(n,N) = \xi_\ell(n)\left[-\log \xi_\ell(n) + 1\right] \\ \xi_\ell = \dfrac{n(n+1)(2n+1)^2}{64\ell^2(\ell+1)^2} + O(\ell^{-6}) \end{cases}$$

$\hookrightarrow$ these can be used to check numerical results.

$R=(n+\frac{1}{2})a$, radius midway between outermost point which was traced over & innermost point which was not



See page 8 of attached reference to see plot of S vs R^2 which I did not have time to reproduce.
The author does this for $N=60$ & $1 \le n \le 30$
Fitting the data gives

$$S = .3\,\mu^2 R^2, \quad \mu = a^{-1}$$

Notes:
The linear behavior cannot ~~contin~~ continue to $n=N$ i.e. we have traced out all d.o.f's!
$\therefore S=0$ $\quad S \to 0$ as $R \to L=(N+1)a$

The wall of our spherical box.

Summary:
- Counting quantum states in a simple setup produced a reduced (Entanglement) entropy proportional to surface area of inaccessible region