No rishon : phase space point of view

* What are rishons?

Rishon model is one of the most famous preon model proposed separately in 1979 by H. Harari and M. Shupe. The model is also called Harari-Shupe model (HSM).
The idea is that all fundamental fermions we know today are composite. There are only two types of truly fundamental fermions : T and V with charge $Q_T = +\frac{1}{3}$ and $Q_V = 0$.
Quarks and leptons (of first generation) can be obtained by combining T and V (and also their antiparticle) in specific ways as follow

ν_e	u_R	u_G	u_B	e^+	$\bar{d}_R$	$\bar{d}_G$	$\bar{d}_B$
VVV	TTV	TVT	VTT	TTT	VVT	VTV	TVV

The corresponding antiparticles : $\bar{\nu}_e, \bar{u}_R, \bar{u}_G, \bar{u}_B, e^-, d_R, d_G, d_B$ are built with V (T) replaced by $\bar{V}(\bar{T})$.

One interesting feature of HSM is that since there is an equal number of protons and electrons in the Universe then HSM tells that actually a number of particles and antiparticles is the same in our Universe. Because

$$\left.\begin{aligned} p &\equiv uud \equiv (2T1V)(2T1V)(2\bar{V}1\bar{T}) \equiv 4T\,1\bar{T}\,2V\,2\bar{V} \\ e^- &\equiv \bar{T}\bar{T}\bar{T} \end{aligned}\right\} \begin{matrix} 6 \text{ rishons} & + & 6 \text{ antirishons} \\ (4T2V) & & (4\bar{T}2\bar{V}) \end{matrix}$$

For the other generations, we can think of them as some kind of excitation state of the 1^{st} generation.

However there are a lot of shortcomings because it cannot explain the following question:

- Why does spin-$\frac{3}{2}$ quarks or leptons not exist?
- Why is the mass of quarks and leptons so small?
- Because there is no rishon-binding dynamics proposed so we also don't know why TTT, VVV are free but TVV, TVV are confined.

However the feature from HSM still holds true that we can see quark & leptons as constructed from two types of charges : $0, \frac{1}{3}$.

	ν_e	u_R	u_G	u_B	e^+	$\bar{d}_R$	$\bar{d}_G$	$\bar{d}_B$
charge :	$0+0+0$	$\frac{1}{3}+\frac{1}{3}+0$	$\frac{1}{3}+0+\frac{1}{3}$	$0+\frac{1}{3}+\frac{1}{3}$	$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$	$0+0+\frac{1}{3}$	$0+\frac{1}{3}+0$	$\frac{1}{3}+0+0$

This feature may not come from the existence of subparticle (rishon) as to be shown in phase space approach. Let us call the above feature as Harari - Shupe observation (HSO).

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* Phase space approach

The motivation of this approach coming from the fact that the symmetry in 3-dimensional macroscopic world somehow relates to quantum numbers characterising elementary particles. For example reflection in 3d position space corresponds to parity. And also the fact that in Hamiltonian mechanics we treat position and momentum variables in almost equal footing in the equation of motion : $\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = \frac{-\partial H}{\partial x}$. The reciprocity transformation where $x \to x' = p$ and $p \to p' = -x$ leaves the equation of motion invariant. This transformation are that take position variable to momentum and vice versa. This might give a hint that a more general symmetry transformation between those two kinds of variables exists and can be related to some new quantum number.

Phase space is a 6 dimensional space of $(\vec{x}, \vec{p})$. Recall that in non-relativistic 3d position (momentum) space, the invariant quantity is $|\vec{x}|^2$ $(|\vec{p}|^2)$. Thus in 6d space, we will assume that the fundamental invariant quantity is $|\vec{x}|^2 + |\vec{p}|^2$

In order for $|\vec{x}|^2$ to be able to add with $|\vec{p}|^2$ we need to introduce a new constant of nature k being of the unit $\frac{[\text{momentum}]}{[\text{position}]}$. Thus the invariance is $k^2|\vec{x}|^2 + |\vec{p}|^2$. We will let $k = 1$ from now on.

Obviously the symmetry group is $O(6)$. Now in order to connect this classical space to quantum world we need to promote x and p to operator satisfied the usual commutation relation

$$[x_m, p_n] = i\delta_{mn} \quad , \quad [x_m, x_n] = 0 \, , \quad [p_m, p_n] = 0 \quad ; \quad \hbar = 1 \, . \qquad — ①$$

We will impose another condition that the symmetry transformation must also preserve ①. Thus the group satisfied both condition is a subgroup of $O(6)$ namely $U(1) \otimes SU(3)$

Before we go into detail on group property, note that our invariant quantity $R^2 \equiv |\vec{x}|^2 + |\vec{p}|^2$ has the same form as Harmonic oscillator Hamiltonian. Thus it is natural to introduce two operators:

$$a_k = \frac{1}{\sqrt{2}}(x_k + ip_k) \quad \text{and} \quad a_k^\dagger = \frac{1}{\sqrt{2}}(x_k - ip_k)$$

Then we can see that the two invariant quantities (or operators) can be written in terms of $a_k, a_k^\dagger$ as

$$|\vec{x}|^2 + |\vec{p}|^2 = \{a_k, a_k^\dagger\} \qquad — ②$$

$$\text{and} \quad -i[\vec{x}, \vec{p}] = [a_k, a_k^\dagger] = 3 \qquad — ③$$

The two operators ②, ③ are quantised. Eigenvalue of ③ is 3 as shown. However eigenvalue of ② can be found by

$$\{a_k, a_k^\dagger\} = [a_k, a_k^\dagger] + 2a_k^\dagger a_k$$

$$= 3 + 2N \qquad \text{where} \quad N \equiv a_k^\dagger a_k \quad \text{(number operator)}$$

thus its eigenvalues are $2N+3$ where $N = 0, 1, 2, \ldots$ We will call the case where $N=0$ as vacuum. Thus the lowest eigenvalue of $|\vec{x}|^2 + |\vec{p}|^2$ is $(|\vec{x}|^2 + |\vec{p}|^2)_{vacuum} = 3$.

Now let us look at properties of the group $U(1)$

We can show that

$$[R^z, x_k] = -2ip_k \quad , \quad [R^z, p_k] = +2ix_k \quad , \quad [R^z, a_k] = -2a_k \quad , \quad [R^z, a_k^\dagger] = +2a_k^\dagger$$

This means that the general transformation is of the form

$$a_k' = \exp\left(i\frac{\phi}{2}R^z\right) a_k \exp\left(-i\frac{\phi}{2}R^z\right) = e^{-i\phi} a_k \qquad — ④$$

$$a_k'^\dagger = \exp\left(i\frac{\phi}{2}R^z\right) a_k^\dagger \exp\left(-i\frac{\phi}{2}R^z\right) = e^{+i\phi} a_k^\dagger$$

where ϕ defined as the common angle of three identical simultaneous rotations in each of the (x_k, p_k) planes. We can see that ④ is a generalized version of the reciprocity transformation which corresponds to the case where $\phi = -\frac{\pi}{2}$.

The other subgroup we mentioned earlier is $SU(3)$

Now comes to the crucial step from which we will show that we can relate the properties of phase space to some quantum numbers of elementary particles.

We exploit the Dirac linearization procedure. In non-relativistic case we have

$$|\vec{p}|^2 = (\vec{p}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma}).$$

This is actually a powerful formula that relates continuous rotations in classical 3D momentum space with the quantum level object (Pauli matrices).

But now we are ready to extend this idea to our phase space. We hope that

$$(\vec{A}\cdot\vec{p} + \vec{B}\cdot\vec{x})(\vec{A}\cdot\vec{p} + \vec{B}\cdot\vec{x}) = |\vec{x}|^2 + |\vec{p}|^2 \qquad — (5)$$

Here A's, B's are matrice to be satisfied special algebra. Consider LHS of (5):

$$\begin{aligned}
(\vec{A}\cdot\vec{p} + \vec{B}\cdot\vec{x})(\vec{A}\cdot\vec{p} + \vec{B}\cdot\vec{x}) &= (A_i p_i + B_i x_i)(A_j p_j + B_j x_j) \\
&= A_i A_j p_i p_j + B_i A_j x_i p_j + A_i B_j p_i x_j + B_i B_j x_i x_j \\
&= \tfrac{1}{2}\Big(A_i A_j p_i p_j + A_i A_j p_i p_j + B_i A_j x_i p_j + B_i A_j x_i p_j + A_i B_j p_i x_j \\
&\qquad + A_i B_j p_i x_j + B_i B_j x_i x_j + B_i B_j x_i x_j \\
&= \tfrac{1}{2}\Big[\{A_i, A_j\} p_i p_j + 2\{A_j, B_i\} x_i p_j - i\,[A_k, B_k] + \{B_i, B_j\} x_i x_j \Big] \quad \text{(for } i \neq j\text{)} \\
&= \tfrac{1}{2}\{A_i, A_j\} p_i p_j + \{A_j, B_i\} x_i p_j - \tfrac{i}{2}[A_k, B_k] + \tfrac{1}{2}\{B_i, B_j\} x_i x_j
\end{aligned}$$

Thus we want $\{A_i, A_j\} = \{B_i, B_j\} = 2\delta_{ij}$ and $\{A_j, B_i\} = 0 \qquad — (6)$

Now
$$\begin{aligned}
(\vec{A}\cdot\vec{p} + \vec{B}\cdot\vec{x})(\vec{A}\cdot\vec{p} + \vec{B}\cdot\vec{x}) &= |\vec{x}|^2 + |\vec{p}|^2 - \tfrac{i}{2}[A_k, B_k] \qquad — (7) \\
&= R^2 + R
\end{aligned}$$

$$R = \sum_k \sigma_k \otimes \sigma_k \otimes \sigma_3$$

Note that $R \equiv -\frac{i}{2}[A_k, B_k]$ arises from the fact that x_i and p_i are not commuted.

In this way (5) will be slightly modified into (7).

We can find representation of matrix A and B satisfied ⑥ in the following from

$$A_i = \sigma_i \otimes \sigma_0 \otimes \sigma_1 \qquad \text{where } \sigma_i\text{'s are Pauli matrices}$$

$$B_i = \sigma_0 \otimes \sigma_i \otimes \sigma_2 \qquad \text{and } \sigma_0 \text{ is } \mathbb{1}_{2\times 2}$$

Now A_i, B_i become 8×8 matrices.

One can also find another matrice which anticommutes with all A_i and B_j as

$$B_7 = i A_1 A_2 A_3 B_1 B_2 B_3 = \sigma_0 \otimes \sigma_0 \otimes \sigma_3$$

Now we try to find what quantum mechanical objects correspond to these matrices.

We propose that the charge operator Q is equivalent to

$$Q \equiv \frac{1}{6}\left[R^2_{vacuum} + R \right] B_7$$

Where $R^2_{vacuum} = 3$ as already mentioned. Thus

$$Q = \frac{1}{2} B_7 + \frac{1}{6} R B_7$$

$$\equiv I_3 + Y/2 \qquad — ⑧$$

where we further identify I_3 as $\frac{B_7}{2} = \frac{1}{2}\sigma_0 \otimes \sigma_0 \otimes \sigma_3$ — ⑨

and Y as $\frac{1}{3} R B_7 = \frac{1}{3}\left(\sum_k \sigma_k \otimes \sigma_k \otimes \sigma_0 \right) \equiv Y_1 + Y_2 + Y_3$ — ⑩

Y_i defined as $\quad Y_i = -\frac{i}{6}[A_i, B_i] B_7 \quad$ (without summing)

It is easy to see that the eigenvalues of I_3 are $\pm\frac{1}{2}$

and the eigenvalues of Y_i are $\pm\frac{1}{3}$. However each Y_i's is not completely independent because there is a relation that

$$Y_1 Y_2 Y_3 = \frac{-1}{27}$$

meaning that either three of them has eigenvalue $-\frac{1}{3}$ or two of them has $+\frac{1}{3}$ and the other has $-\frac{1}{3}$.

Then the possible eigenvalues of Y are

$$Y = -\tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{3} = -1 \qquad \text{or} \qquad Y = +\tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{3} = \tfrac{1}{3}$$

We can see that for the latter case we also can write

$$Y = +\tfrac{1}{3} - \tfrac{1}{3} + \tfrac{1}{3} \qquad \text{or} \qquad Y = -\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3}.$$

This looks similar to what we have in HSO. So it is reasonable to propose that rishons in HSM are equivalent to our Y_i where T corresponds to $Y_i = \frac{1}{3}$ and V to $Y_i = -\frac{1}{3}$. The diagram now is

	ν_e	u_R	u_G	u_B	e^+	$\bar{d}_R$	$\bar{d}_G$	$\bar{d}_B$	
charge :	$0+0+0$	$\frac{1}{3}+\frac{1}{3}+0$	$\frac{1}{3}+0+\frac{1}{3}$	$0+\frac{1}{3}+\frac{1}{3}$	$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$	$0+0+\frac{1}{3}$	$0+\frac{1}{3}+0$	$\frac{1}{3}+0+0$	– a.)
Rishon	VVV	TTV	TVT	VTT	TTT	VVT	VTV	TVV	– b.)
$\frac{Y_i}{2}$	$-\frac{1}{6}\ -\frac{1}{6}\ -\frac{1}{6}$	$+\frac{1}{6}\ +\frac{1}{6}\ -\frac{1}{6}$	$+\frac{1}{6}\ -\frac{1}{6}\ +\frac{1}{6}$	$-\frac{1}{6}\ +\frac{1}{6}\ +\frac{1}{6}$	$+\frac{1}{6}\ +\frac{1}{6}\ +\frac{1}{6}$	$-\frac{1}{6}\ -\frac{1}{6}\ +\frac{1}{6}$	$-\frac{1}{6}\ +\frac{1}{6}\ -\frac{1}{6}$	$+\frac{1}{6}\ -\frac{1}{6}\ -\frac{1}{6}$	– c.)

One notes strict correspondence between a.) and c.). a.) is simply obtained from c.) by adding $+\frac{1}{6}$ to each eigenvalue of $\frac{Y_i}{2}$. Thus the phase space approach indicates that the original HSO could be equally well formulated in terms of the eigenvalues of Y and Y_i without the need of subparticles.