Three dimensional Chern-Simons theory

1. The Chern-Simons 3-form

The general Chern-Simons $(2n-1)$-form is defined as

$$d W_{cs}^{(2n-1)} = Tr(F^n)$$

where the curvature 2-form F is given by

$$F = dA + A \wedge A$$

with A being the Lie algebra valued connection 1-form

$$A = A_\mu dx^\mu = A_\mu^a dx^\mu \otimes T^a \quad , \quad T^a \in \text{Lie } G$$

and $\{x^\mu\}$ being the local coordinates.

① graded cyclic property of the trace: suppose α and β are Lie algebra valued p-form and q-form respectively, then

$$Tr(\alpha \wedge \beta) = (-1)^{pq} Tr(\beta \wedge \alpha)$$

② trace commutes with exterior derivative:

$$Tr(d\alpha) = d(Tr\alpha)$$

Scanned by CamScanner

Consider $n=2$ case,

$$\mathrm{Tr}(F^2) = \mathrm{Tr}[(dA + A\wedge A)\wedge(dA + A\wedge A)]$$

$$= \mathrm{Tr}(dA\wedge dA + A\wedge A\wedge dA + dA\wedge A\wedge A + A\wedge A\wedge A\wedge A)$$

$$= \mathrm{Tr}(dA\wedge dA + 2A\wedge A\wedge dA)$$

$$= d\,\mathrm{Tr}(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A)$$

where $\mathrm{Tr}(A\wedge A\wedge A\wedge A) = 0$, since

$$\mathrm{Tr}[A\wedge(A\wedge A\wedge A)] = (-1)^{1\times 3}\,\mathrm{Tr}[(A\wedge A\wedge A)\wedge A]$$

So, by definition, the Chern-Simons 3-form is

$$\omega_{CS}^{(3)} = \mathrm{Tr}(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A)$$

2. Chern-Simons Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\int_{M^3}\mathrm{Tr}(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A)$$

where M^3 is a three dimensional manifold.

Scanned by CamScanner

Choose a local coordinates $\{x^\mu\}$, then

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\int_{M^3} \mathrm{Tr}\Big[A_\mu dx^\mu \wedge (\partial_\nu A_\rho dx^\nu \wedge dx^\rho) + \frac{2}{3}(A_\mu dx^\mu)\wedge(A_\nu dx^\nu)\wedge(A_\rho dx^\rho)\Big]$$

$$= \frac{k}{4\pi}\int_{M^3} \mathrm{Tr}\Big[A_\mu \tfrac{1}{2}(\partial_\nu A_\rho - \partial_\rho A_\nu) dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{3} A_\mu (A_\nu A_\rho - A_\rho A_\nu) dx^\mu \wedge dx^\nu \wedge dx^\rho\Big]$$

$$= \frac{k}{8\pi}\int_{M^3} \varepsilon^{\mu\nu\rho}\, \mathrm{Tr}\Big(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \tfrac{2}{3} A_\mu [A_\nu, A_\rho]\Big)$$

where $dx^\mu \wedge dx^\nu \wedge dx^\rho = \varepsilon^{\mu\nu\rho}\,\omega_{vol}$.

3. Gauge transformation of $\mathcal{L}_{CS}$

Under the map $g: M \to G$,

$$A \mapsto A^g = g^{-1}(d+A)g = g^{-1}Ag + g^{-1}dg$$

then

$$\mathcal{L}_{CS}[A^g] = \frac{k}{4\pi}\int_{M^3} \mathrm{Tr}\Big[(g^{-1}Ag + g^{-1}dg)\wedge d(g^{-1}Ag + g^{-1}dg) + \frac{2}{3}(g^{-1}Ag + g^{-1}dg)\wedge(g^{-1}Ag + g^{-1}dg)\wedge(g^{-1}Ag + g^{-1}dg)\Big]$$

Scanned by CamScanner

Define the Chern-Simons invariant as

$$I(A) = \frac{1}{4\pi}\int_{M^3} \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

then

$$\mathcal{L}_{cs}[A^g] = kI(A) + \frac{k}{4\pi}\int_{M^3} \mathrm{Tr}\, d(A\wedge g^{-1}dg)$$
$$+ \frac{k}{4\pi}\cdot\frac{1}{3}\int_{M^3} \mathrm{Tr}\,(g^{-1}dg)^{\wedge 3}$$

the second term of above equation is a total derivative, which can be dropped by imposing vanishing boundary condition.

The winding number of the map $g: M \to G$ is given by

$$m = \frac{1}{24\pi^2}\int_{M^3} \mathrm{Tr}\,(g^{-1}dg)^{\wedge 3}$$

So, under the gauge transformation,

$$\mathcal{L}_{cs}[A^g] = \mathcal{L}_{cs}[A] + 2\pi k\cdot m$$

it shows that $\mathcal{L}_{cs}[A]$ itself is not gauge invariant.

Scanned by CamScanner

For a consistent quantum theory, it does only require the single-valuedness of $\exp(i\mathcal{L})$.

So in this case, k should be an integer, that is $k \in \mathbb{Z}$. k is called the level of Chern-Simons theory.

4. Classical solutions of $\mathcal{L}_{CS}$

By applying variation principle,

$$\delta \mathcal{L}_{CS}[A] = \frac{k}{4\pi} \delta \int_{M^3} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

$$= \frac{k}{4\pi} \int_{M^3} \mathrm{Tr}\left(\delta A \wedge dA + A \wedge d\delta A + 2\delta A \wedge A \wedge A\right)$$

since $d(\delta A \wedge A) = d\delta A \wedge A - \delta A \wedge dA$

then $\mathrm{Tr}[d(\delta A \wedge A)] = \mathrm{Tr}(d\delta A \wedge A) - \mathrm{Tr}(\delta A \wedge dA)$

$$= \mathrm{Tr}(A \wedge d\delta A) - \mathrm{Tr}(\delta A \wedge dA)$$

$$\Rightarrow \mathrm{Tr}(A \wedge d\delta A) = \mathrm{Tr}[d(\delta A \wedge A)] + \mathrm{Tr}(\delta A \wedge dA)$$

Scanned by CamScanner

So, we have

$$\delta \mathcal{L}_{cs}[A] = \frac{k}{4\pi}\int_{M^3} \mathrm{Tr}[\cancel{d(\delta A\wedge A)} + 2\delta A\wedge dA + 2\delta A\wedge A\wedge A]$$

$$= \frac{k}{2\pi}\int_{M^3} \mathrm{Tr}[\delta A\wedge(dA + A\wedge A)]$$

$$= \frac{k}{2\pi}\int_{M^3} \mathrm{Tr}[\delta A\wedge F] = 0 \implies F = 0$$

the classical solutions are flat connections, which is corresponding to $F = DA = dA + A\wedge A = 0$

5. Large k limit (weak coupling)

Define the coupling as $g^2 = \frac{4\pi}{k}$, large k corresponds to weak coupling.

For large k, the partition function is

$$Z = \int DA \exp(i\mathcal{L}_{cs})$$

$$= \int DA \exp\left(\frac{ik}{4\pi}\int_{M^3} \mathrm{Tr}(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A)\right)$$

Scanned by CamScanner

Since for large k, the integral oscillates wildly, the main contribution comes from stationary points, which are the classical solutions $\rightarrow$ flat connections $A^{(0)}$.

Take the stationary phase approximation, we have

$$Z = \sum_k Z(A^{(0)k})$$

with a complete set of gauge equivalence classes of flat connections $\{A^{(0)k}\}$.

Set $A = A^{(0)} + B$, the Chern-Simons invariant is

$$I(A) = I(A^{(0)} + B)$$

$$= \frac{1}{4\pi}\int_{M^3} \mathrm{Tr}\left[(A^{(0)}+B)\wedge d(A^{(0)}+B) + \frac{2}{3}(A^{(0)}+B)^{\wedge 3}\right]$$

$$= I(A^{(0)}) + \frac{1}{4\pi}\int_{M^3} \mathrm{Tr}\left[B\wedge dB + B\wedge dA^{(0)} + A^{(0)}\wedge dB + 2B\wedge A^{(0)}\wedge A^{(0)} + 2B\wedge A^{(0)}\wedge B + \frac{2}{3}\cancel{B\wedge B\wedge B}\right]$$

where we ignore the $B\wedge B\wedge B$ term.

Scanned by CamScanner

$$\because\ d(A^{(0)}\wedge B) = dA^{(0)}\wedge B - A^{(0)}\wedge dB$$

$$\mathrm{Tr}\, d(A^{(0)}\wedge B) = \mathrm{Tr}(dA^{(0)}\wedge B) - \mathrm{Tr}(A^{(0)}\wedge dB)$$

$$= \mathrm{Tr}(B\wedge dA^{(0)}) - \mathrm{Tr}(A^{(0)}\wedge dB)$$

$$\therefore\ \mathrm{Tr}(A^{(0)}\wedge dB) = \mathrm{Tr}\, d(A^{(0)}\wedge B) + \mathrm{Tr}(B\wedge dA^{(0)})$$

$$\therefore\ I(A) = I(A^{0}) + \frac{1}{4\pi}\int_{M^3} \mathrm{Tr}\left[2B\wedge(\cancel{dA^{(0)}} + A^{(0)}\wedge A^{(0)}) + B\wedge D^{(0)}B\right]$$

$$= I(A^{0}) + \frac{1}{4\pi}\int_{M^3} \mathrm{Tr}(B\wedge D^{(0)}B)$$

where $D^{(0)}B$ is the exterior covariant derivative of B with respect to flat connection $A^{(0)}$, which is not depending on a metric of M^3.

Then we have

$$\mathcal{L}_{CS} = k\, I(A^{(0)}) + \frac{k}{4\pi}\int_{M^3} \mathrm{Tr}(B\wedge D^{(0)}B)$$

Scanned by CamScanner

6. Gauge fixing

To perform the integral, one has to fix the gauge.
It needs to pick a metric on M. With picking a metric, we choose Lorentz gauge $D^{(0)}_\mu B^\mu = 0$, ($D^{(0)}_\mu$ is relevant to the metric of M and flat connection (background gauge field) $A^{(0)}$). Introduce the Faddeev-Popov gauge fixing term,

$$\mathcal{L}_{gf} = \frac{k}{4\pi}\int_{M^3} \left(\mathrm{Tr}\,\phi\, D^{(0)}_\mu B^\mu + \mathrm{Tr}\,\bar{c}\, D^{(0)}_\mu D^{(0)\mu} c\right)$$

where B and ϕ are Lie algebra valued 1-form and 3-form.
Then the partition function becomes

$$Z(A^{(0)}) = \int DB D\phi \exp\left[i\left(\mathcal{L}_{cs} + \mathcal{L}_{gf}\right)\right]$$

$$= \int DB D\phi \, Dc D\bar{c} \exp\Big[ikI(A^{(0)}) + \frac{ik}{4\pi}\int_{M^3} \mathrm{Tr}(B\wedge D^{(0)}B)$$

$$+ \frac{ik}{4\pi}\int_{M^3}\left(\mathrm{Tr}\,\phi D^{(0)}_\mu B^\mu + \mathrm{Tr}\,\bar{c} D^{(0)}_\mu D^{(0)\mu} c\right)\Big]$$

$$= e^{ikI(A^{(0)})} \int DB D\phi \, Dc D\bar{c} \exp\Big[\frac{ik}{4\pi}\int_{M^3} \mathrm{Tr}\,(B\wedge D^{(0)}B + \phi D^{(0)}_\mu B^\mu$$

$$+ \bar{c} D^{(0)}_\mu D^{(0)\mu} c)\Big]$$

Scanned by CamScanner

The c and $\bar{c}$ make the correct integral measure on the space of $\mathcal{A}/\mathcal{G}$. Define the Hodge star as

$$* : \Lambda^k(M) \longrightarrow \Lambda^{3-k}(M)$$

and the operator $L = *D^{(0)} + D^{(0)}*$

$$L : \Lambda^k(M) \longrightarrow \Lambda^k(M)$$

Introduce the restriction of L as

$$L_- : \Lambda^{odd}(M) \longrightarrow \Lambda^{odd}(M)$$

Then we have

$$Z(A^{(0)}) = e^{ikI(A^{(0)})} \cdot \frac{\det(D^{(0)}_\mu D^{(0)\mu})}{\sqrt{\det(L_-)}}$$

where the ghost determinant of Laplacian $D^{(0)}_\mu D^{(0)\mu}$ is real and positive. So,

$$Z(A^{(0)}) = e^{ikI(A^{(0)})} \frac{\det(D^{(0)}_\mu D^{(0)\mu})}{\left|\sqrt{\det(L_-)}\right|} e^{i\varphi}$$

$$= e^{ikI(A^{(0)})} T(A^{(0)}) e^{i\varphi}$$

where $T(A^{(0)})$ is the Ray-Singer torsion of $A^{(0)}$, which is a topological invariant.

Scanned by CamScanner

7. The phase of $(\det^{-\frac{1}{2}} L_-)$

consider the integral

$$\int DBD\phi \exp\left(i\int_{M^3} Tr(B\wedge DB+\phi D*B)\right)$$

assume that $L_-\Psi_i=\lambda_i\Psi_i$, then we have

$$\Rightarrow \prod_i \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{i\lambda_i\Psi_i^2} d\Psi_i$$

$$= \prod_i \frac{1}{|\sqrt{\lambda_i}|} \exp\left(\frac{i\pi}{4}\operatorname{sign}\lambda_i\right)$$

Introduce the eta invariant of $A^{(0)}$, which is

$$\eta(A^{(0)}) = \frac{1}{2}\lim_{s\to 0}\sum_i \operatorname{sign}\lambda_i |\lambda_i|^{-s}$$

then we have

$$\det^{-\frac{1}{2}} L_- = |\det^{-\frac{1}{2}} L_-| \cdot e^{i\frac{\pi}{2}\eta(A^{(0)})}$$

By using the Atiyah-Patodi-Singer theorem,

$$\frac{1}{2}(\eta(A^{(0)})-\eta(0)) = \frac{C(G)}{2\pi} I(A^{(0)})$$

where $\eta(0)$ is η-invariant of $A=0$, $C(G)$ is the Casimir operator in the adjoint representation of G, $C(G)=(T^a_{adj})(T^a_{adj})$

Scanned by CamScanner

The partition function now becomes

$$Z = \sum_k Z(A^{(0)k})$$

$$= e^{i\frac{\pi}{2}\eta(0)} \sum_k e^{i(k+\frac{1}{2}C(G))I(A^{(0)k})} \cdot T(A^{(0)k})$$

where $\eta(0)$ is not topological invariant.

8. Gravitational Chern-Simons counterterm

$\eta(0)$ is the η invariant of L_- with some metric of M and trivial $A=0$. L_- consists of d copies of L_-^{grav}, L_-^{grav} only depends on the metric, where d = dimension of G. Then we have

$$\eta(0) = d \cdot \eta_g(\omega)$$

So the phase factor is $\exp\left(\frac{i d\pi}{2}\eta_g(\omega)\right)$.

Consider the gravitational Chern-Simons term,

$$I(\omega) = \frac{1}{4\pi}\int_{M^3} \mathrm{Tr}\left(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right)$$

Scanned by CamScanner

Relate $I(w)$ to the tangent bundle of M.

$I(w)$ will differ with different trivialization of tangent bundle of M. If the trivialization of TM differs by s units,

$$I(w) \to I(w) + 2\pi s \,, \quad s \text{ is an integer}$$

by the Atiyah-Patodi-Singer theorem,

$$\frac{1}{2}\eta_g(w) + \frac{1}{12}\cdot\frac{I(w)}{2\pi}$$

is a topological invariant, depending on a choice of trivialization of TM, not on metric of M.

In other words, it depends on a choice of framing of oriented 3-manifold M.

For large k limit, the topological invariant partition function of three dimension Chern-Simons theory is

$$Z = \exp\left(i\pi d\left(\frac{\eta_g(w)}{2} + \frac{I(w)}{24\pi}\right)\right)\cdot\sum_k e^{i(k+C(G)/2)I(A^{(0)k})}\cdot T(A^{(0)k})$$

If the framing is shifted by s units,

$$Z \to Z\cdot\exp\left(i2\pi s\cdot\frac{d}{24}\right)$$

Scanned by CamScanner