HEP Theory Talk Plan

1) Intro

2) Unruh - De Witt Detector

3) Bogoliubov Transformation

4) Rindler Space and Unruh Effect

5) Hawking Radiation

1) Introduction

Last talks all about entropy and the area law.
The basic statement that was lingering in the background all the time was that the Hawking temperature of a Black Hole is

$$T_H = \frac{\hbar c^3}{8\pi G k_B M} \frac{\kappa}{2\pi} \quad \text{and the corresponding entropy} \quad S_{BH} = \frac{c^3 A}{4G\hbar} \frac{A}{4} \propto A$$

$$dE = TdS - pdV \qquad\qquad \delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J$$

Focus today:
- Temperature of BH
- More generally: QFT in curved space and how the particle concept fails there.

Start with
- Practical example: Unruh-De Witt Detector: accelerated detector does see the Minkowski vacuum as having particles and a temperature: Unruh Effect

OUTLINE
- Bogoliubov Transformation: Introduction
- Rindler space → again same result for Unruh Effect
- Hawking Temperature

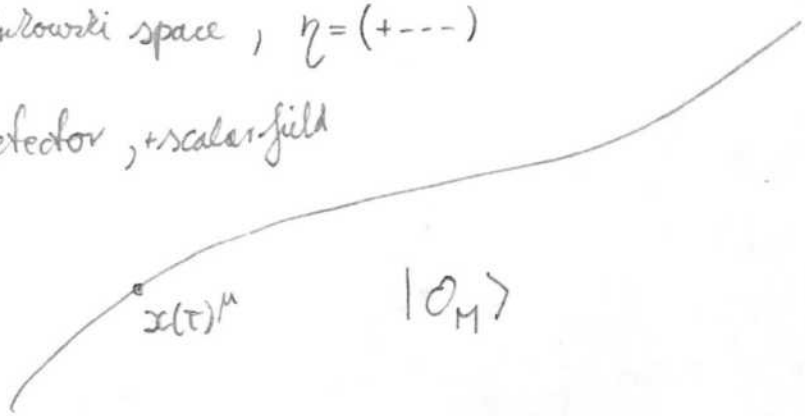
2) Unruh De Witt Detector

We will see how the motion of the detector can affect if we measure particles

Birrell Davies
QFT in curved space

Setting : - Minkowski space , $\eta = (+---)$

- Detector , + scalar field



	Ground State		Excited State
ϕ :	$\lvert 0_M\rangle$	$\xrightarrow{\text{interact}}$	$\lvert\psi\rangle$
detector :	$\lvert E_0\rangle$		$\lvert E\rangle$

$$\mathcal{L}_{int} = c\, m(\tau)\, \phi(x(\tau))$$ local point interaction

$m(\tau)$: monopole moment

Transition amplitude to first order

interaction picture and use $H_0|E\rangle = E|E\rangle$

$$\mathcal{A} \approx \langle E, \psi | ic \int_{-\infty}^{\infty} d\tau\, m(\tau)\, \phi(x(\tau)) | 0_M, E\rangle$$

E: detector, ψ: field, $m(\tau)$: $e^{iH_0\tau} m(0) e^{-iH_0\tau}$ time evolution of $m(\tau)$

$$= ic \langle E | m(0) | E_0\rangle \int_{-\infty}^{\infty} d\tau\, e^{i\underbrace{(E-E_0)}_{\Delta E}\tau} \langle \psi | \phi(x) | 0_M\rangle$$

↓ not like in book from here

ϕ real scalar field

$$\gamma_\psi = |\mathcal{A}|^2 = \mathcal{A}^*\mathcal{A} = c^2 |\langle E|m(0)|E_0\rangle| \left[\int_{-\infty}^{\infty} d\tau\, e^{-i\Delta E\tau} \langle 0_M|\phi(x)|\psi\rangle\right]\left[\int_{-\infty}^{\infty} d\tau'\, e^{i\Delta E\tau'} \langle\psi|\phi(x')|0_M\rangle\right]$$

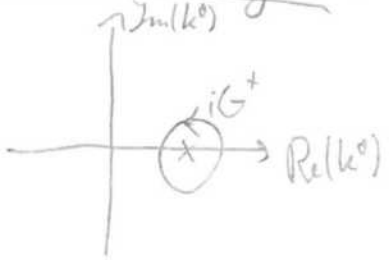
$$= c^2 |\langle \ldots\rangle|^2 \int_{-\infty}^{\infty} d\tau\, d\tau'\, e^{-i\Delta E(\tau-\tau')} \langle 0_M|\phi(x)|\psi\rangle\langle\psi|\phi(x')|0_M\rangle$$

Transition to all possible states

$$\Gamma_E = \sum_\psi \gamma_\psi = c^2 |\langle E|m(0)|E_0\rangle| \int d\tau\, d\tau'\, e^{-i\Delta E(\tau-\tau')} \underbrace{\langle 0_M|\phi(x(\tau))\,\phi(x(\tau'))|0_M\rangle}_{G^+(x,x')}$$

$G^+(x,x')$ pos. freq. Wightman function

<u>massless scalar field</u>: $i G^+(x,x') = \frac{1}{(4\pi)^4} \int d^3\vec{k}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, 2\pi i \operatorname{Res}_{\sqrt{\vec{k}^2}} \left[\frac{e^{-ik^0(t-t')}}{(k^0)^2 - \vec{k}^2} \right]$



Won't calculate G^+ here explicitly... it is

$$\boxed{i G^+(x-x') = \frac{i}{(2\pi)^2} \frac{1}{(t-t'-i\epsilon)^2 - |x-x'|^2}}$$

<u>(I) Inertial observer</u>: $\vec{v} = \text{const}$: Go back to $\mathcal{A}$ $\qquad \phi(x) = \int d^3k' \frac{1}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left[a_{\vec{k}} e^{i\vec{k}\cdot\vec{x} - i\omega t} + a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x} - i\omega t} \right]$

$$\langle \psi | \phi(x) | 0_M \rangle = \langle 1_{\vec{k}} | \phi(x) | 0_M \rangle = \int d^3k' \frac{1}{\sqrt{(2\pi)^3 2\omega'}} \underbrace{\langle 1_{\vec{k}} | a^\dagger_{\vec{k}'} | 0_M \rangle}_{\delta(\vec{k}-\vec{k}')} e^{-i\vec{k}'\cdot\vec{x} + i\omega' t}$$

(we sum over all ψ anyways!)

$$\vec{x}(\tau) = \vec{x}_0 + \vec{v} t = \vec{x}_0 + \frac{\vec{v}\tau}{\sqrt{1-\vec{v}^2}}$$

$$\mathcal{A} = i c \langle \cdot/\cdot \rangle \int d\tau\, e^{i(E-E_0)\tau} \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-i\vec{k}\cdot\vec{x}_0} e^{-i\vec{k}\cdot\vec{v}\tau/\sqrt{1-v^2}} e^{i\omega\tau/\sqrt{1-v^2}}$$

$$= i c \langle \cdot/\cdot \rangle \frac{1}{\sqrt{(2\pi)^3 2\omega}} e^{-i\vec{k}\cdot\vec{x}_0} (2\pi)\, \delta\Big(\overbrace{E - E_0 + \frac{\omega - \vec{k}\cdot\vec{v}}{\sqrt{1-\vec{v}^2}}}^{>0 \text{ always}} \Big) = 0$$

Basically just Energy conservation

Poincaré invariance

$$\vec{k}\cdot\vec{v} \le |\vec{k}|\underbrace{|\vec{v}|}_{\le 1} < \omega \quad , \omega = \sqrt{\vec{k}^2 + m^2}$$

$$E > E_0$$

<u>(II) Observer with const. acceleration</u>

Minkowski $G^+(x,x') = G^+(x-x')$
$= G(x(\tau) - x(\tau'))$
$= G(\tau - \tau')$
We get $\tilde{G}(E)(\int d\tau)$

$$\Gamma_E = c^2 \underbrace{|\langle E | m(0) | E_0 \rangle|^2}_{\text{selectivity of detector, only depends on how it is built}} \underbrace{\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau'\, e^{-i\Delta E(\tau-\tau')} G^+(x-x')}_{\mathcal{F}(E) \text{ "detector response function" indep of how it is built.}}$$

- divide by $(\int d\tau)$ to get rate
- Accel. observer $x=y=0$, $z(t)^2 = t^2 - \alpha^2$, $z(\tau) = \alpha \cosh(\frac{\tau}{\alpha})$, $\frac{1}{\alpha} = a$ accel.
- $G^+(\Delta\tau) = \frac{1}{16\pi^2\alpha^2} \frac{1}{\sinh^2(i\frac{\Delta\tau}{2\alpha} - \frac{\epsilon}{\alpha})}$ LONG CALCULATION
- $\boxed{\dot{\Gamma}_E = \frac{c^2}{(2\pi)^2} |\langle \cdot/\cdot \rangle| \frac{(E-E_0)|\langle E | m(0) | E_0 \rangle|^2}{e^{2\pi(E-E_0)\alpha} - 1}}$ | Planck $\frac{1}{e^{\Delta E/T} - 1} \Rightarrow \boxed{T = \frac{1}{2\pi\alpha} = \frac{a}{2\pi}}$

$\dot{\Gamma} = \sum_E \Gamma_E$ Planck factor $\frac{1}{e^{2\pi(E-E_0)/a}-1}$

Equilibrium between ϕ and accel. detector is same as would be if unaccel., but thermal bath

Uniformly accelerated observer sees thermal bath of $T = \frac{a}{2\pi}$ eventhough ϕ is in state $|0_M\rangle$.

Inertial observer: No particles

$|E_0\rangle \to |E\rangle$, $|0_M\rangle \to |\psi\rangle = |1_{\vec{k}}\rangle$

Detector and field gain energy

E comes from actor that does the accel. → "friction".

Why don't we see the Unruh effect?

CERN

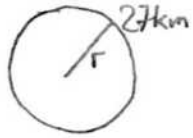
$r \sim 4$ km

$T = \frac{a}{\pi}$

Why don't we see Unruh radiation?

maybe leave out

Say: CERN



$\overset{\sim 6}{2\pi} r \sim 27\,km$

$r \sim 4\,km$

$$a = \frac{v^2}{r} \sim \frac{c^2}{r}$$

$$T = \frac{\hbar a}{2\pi c k_B} \sim \frac{\hbar c^2}{2\pi c k_B r} \sim \frac{\cancel{6}\cdot 10^{-16}\,eVs \cdot 3\cdot 10^8\,m/s}{\cancel{2\pi}\cdot 9\cdot 10^{-5}\frac{eV}{K}\cdot r} \sim \frac{10^{-3}\,m}{3r}\,K \sim \frac{10^{-3}\,m\,K}{3\cdot 4\cdot 10^3\,m} \sim 10^{-7}\,K$$

for $T \sim 1K$:

$$1K \sim \frac{10^{-3}}{3}K\,\frac{1m}{r} \Rightarrow r \sim \frac{3}{10^{-3}}\,m \sim 3\,mm$$

Best definition for vacuum state?

- we saw that what is vacuum for one observer might not be vacuum for another observer

  Minkowski – Rindler (accelerated)

- $|0_M\rangle$ Minkowski is "strong candidate" for the "correct" vacuum

  inertial observer is special

- In curved space (grav. field): Inertial → free falling

  ↳ free falling observers generally will not ~~see same vacuum~~ agree on a choice of vac.

3) Bogoliubov Transformation

Just idea/summary. More in A1, A2

Transformation to other observer can mix the creation and annihilation operators

$u, \bar{u}$ complete sets

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right], \quad |0\rangle, \quad a_j|0\rangle = 0 \quad \forall j$$

$$= \sum_j \left[\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x) \right], \quad |\bar{0}\rangle, \quad \bar{a}_j|\bar{0}\rangle = 0 \quad \forall j$$

Minkowski: Natural set of modes connected to (x, y, z, t), basically Poincaré group

∂_t : Killing vector, $\perp t=0$ hypersurf.

$\partial_t u(x) = -i\omega u(x)$, $\omega > 0$ pos. freq. modes

Curved space: No poincare symm. $\quad (\phi_1, \phi_2) = -i \int_\Sigma \phi_1 \overleftrightarrow{\partial}_\mu \phi_2^* \sqrt{g_\Sigma}\, d\Sigma^\mu$

$$\bar{u}_j = \sum_i \left[\alpha_{ji} u_i + \beta_{ji} u_i^* \right], \quad \bar{u}_j^* = \sum_i \alpha_{ji}^* u_i^* + \beta_{ji}^* u_i$$

$$u_i = \sum_j \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^* \right)$$

$$\phi(x) = \sum_i \left[a_i u_i(x) + a^\dagger u_i^*(x) \right] = \sum_j \left\{ \underbrace{\sum_i \left(a_i \alpha_{ji}^* + a_i^\dagger \beta_{ji}^* \right)}_{\bar{a}_j} \bar{u}_j + c.c. \right\}$$

$$\boxed{\bar{a}_j = \sum_i \left(a_i \alpha_{ji}^* - a_i^\dagger \beta_{ji}^* \right)}$$

$\bar{a}_j^\dagger$ analogue and a_i, $a_i^\dagger$ too

$$\boxed{\langle \bar{0} | N_i | \bar{0} \rangle} = \langle \bar{0} | a_i^\dagger a_i | \bar{0} \rangle = \boxed{\sum_j |\beta_{ji}|^2}$$

What one observer sees as vacuum contains particles for other observer if $\beta \neq 0$

QFT in Rindler Space (BD), Carroll

$$ds^2 = -dt^2 + dx^2$$

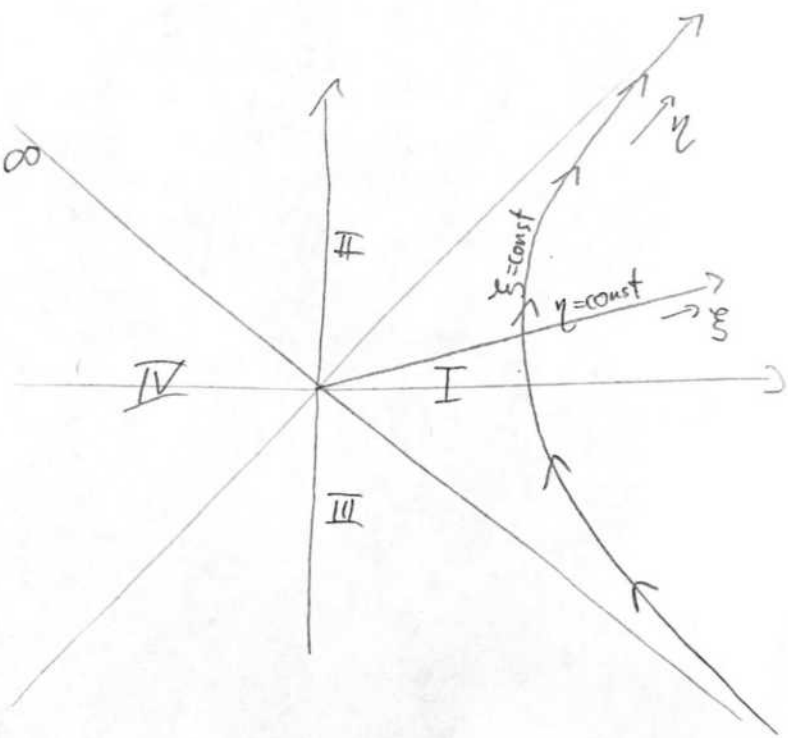
$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau) \qquad x^2 - t^2 = \frac{1}{\alpha^2}$$

$$x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau)$$

$$a^\mu = \frac{d^2x^\mu}{d\tau^2}, \quad a_\mu a^\mu = \alpha^2 \text{ proper accel.}$$

uniformly accel. observer

Describe (portion of) spacetime with coords. of accel. observer



Rindler Space $\eta, \xi \quad -\infty < \eta, \xi < \infty$

$$t = \frac{1}{a} e^{a\xi}\sinh(a\eta)$$
$$x = \frac{1}{a} e^{a\xi}\cosh(a\eta)$$

only $x > |t|$

$$x^2 - t^2 = \underbrace{\frac{1}{a^2} e^{2a\xi}}_{\frac{1}{\alpha^2}} \Rightarrow \alpha = \frac{1}{a} e^{a\xi}$$

$$ds^2 = e^{2a\xi}\left(-d\eta^2 + d\xi^2\right)$$

- Modes in I and IV to describe all ϕ on $t=0$ $\quad g_k^{(1)}, g_k^{(2)}, \hat{b}_k^{(1)}, \hat{b}_k^{(2)}$
- Minkowski: $f_k, \hat{a}_k$

→ Bogoliubov transformation

- $\langle 0_M | b_k^{(1)} b_k^{(1)} | 0_M \rangle = \frac{1}{e^{2\pi\omega/a} - 1}\delta(0)$

$$T = \frac{a}{2\pi}$$

Mink	Bogoliubov	Rindler
$f_k, \hat{a}_k$		$g_k^{(1)}, g_k^{(2)}, \hat{b}_k^{(1)}, \hat{b}_k^{(2)}$
$a_k \lvert 0_M \rangle$		$\hat{b}_k^{(1)} \lvert 0_R \rangle = \hat{b}_k^{(2)} \lvert 0_L \rangle = 0$

Hawking Radiation

Sources: • Zee, "Einst. Gravity in Nutshell"
• BD
• Carroll
• Wiki: Hawking Rad.

Rindler:

$$g = -dt^2 + dr^2 + r^2 d\Omega^2 \quad \Big| \quad \begin{array}{l} t = \rho \sinh T \\ r = \rho \cosh T \end{array} \qquad a = \frac{1}{\rho}$$

$$= -\rho^2 dT^2 + d\rho^2 + \rho^2 \cosh^2 T \, d\Omega^2$$

$\boxed{\text{Details in Appendix A8}}$

Schwarzschild

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\cancel{r}} dr^2 + r^2 d\Omega^2 \quad \Big| \quad \text{Near horizon } \boxed{r = 2M + \frac{\rho^2}{8M}}$$

$$= -\left(1 - \frac{2M}{2M + \frac{\rho^2}{8M}}\right) dt^2 + \frac{1}{\cancel{r}} dr^2 + \left(2M + \frac{\rho^2}{8M}\right)^2 d\Omega^2 \quad \Big| \quad dr = \frac{\rho}{4M} d\rho$$

$$\approx -\left(1 - 1 + \frac{\rho^2}{16M}\right) dt^2 + \frac{1}{\cancel{r}} \left(\frac{\rho}{4M}\right)^2 d\rho^2 + (2M)^2 d\Omega^2$$

$$= -\left(\frac{\rho}{4M}\right)^2 dt^2 + d\rho^2 + 2M \, d\Omega^2 \quad \Big| \quad \boxed{T = \frac{t}{4M}}$$

$$= -\rho^2 dT^2 + d\rho^2 + (2M)^2 d\Omega^2$$

for $d\Omega^2 = 0$

Rindler $\approx$ Schwarzschild (near horizon)

$\boxed{a = \frac{1}{\rho}}$ proper accel.

$a_{\text{Horizon}} \to \infty \Rightarrow$ Need to look at a_{Horizon} as seen from ∞

$$a_r = \frac{1}{\rho}$$

$$a_\infty = \underbrace{\sqrt{\frac{g_{00}(r)}{g_{00}(\infty)}}}_{\text{redshift factor}} a_r \Rightarrow \alpha = a_\infty(r \to r_H) = \lim_{\rho \to 0} \sqrt{\frac{1 - \frac{2M}{r}}{1}} \frac{1}{\rho}$$

$$= \lim_{\rho \to 0} \sqrt{\frac{r - 2M}{r}} \frac{1}{\rho} = \lim_{\rho \to 0} \sqrt{\frac{\frac{\rho^2}{8M}}{2M + \frac{\rho^2}{8M}}} \frac{1}{\rho}$$

$$= \lim_{\rho \to 0} \sqrt{\frac{1}{16M^2}} = \frac{1}{4M} \quad \Big| \quad \boxed{T = \frac{a_\infty}{2\pi} = \frac{1}{8\pi M}}$$

$T = \frac{1}{8M}$, corresponding entropy $S = \frac{A}{4}$

Entropy counts microstates

↳ What are the microstates of a BH?

active research

Birrell, Davies 3.2 Scalar-field quantization

09/19/19

$\xi = 0$ minimally coupled

$\xi = \frac{1}{4}\frac{n-2}{n-1}$ conformally coupled

$$\mathcal{L}(x) = \frac{1}{2}\sqrt{-g}\left\{ g^{\mu\nu}(x)\,\phi(x)_{,\mu}\phi(x)_{,\nu} - [m^2 + \xi R(x)]\phi^2(x)\right\}$$

$$S = \int d^n x\, \mathcal{L}(x)$$

$$(\phi_1, \phi_2) = -i\int_\Sigma \phi_1(x)\overleftrightarrow{\partial}_\mu \phi_2^*(x)\sqrt{-g_2(x)}\, \overset{n^\mu d\Sigma}{d\Sigma^\mu} = -i\int_\Sigma \left\{\phi_1(x)\partial_\mu\phi_2^*(x) - [\partial_\mu\phi_1(x)]\phi_2(x)^*\right\}\sqrt{-g_2(x)}\, d\Sigma^\mu$$

$u_i(x)$, $(u_i, u_j) = \delta_{ij}$, $(u_i^*, u_j^*) = -\delta_{ij}$, $(u_i, u_j^*) = 0$

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x)\right], \quad [a_i, a_j^\dagger] = \delta_{ij}$$

leaves minkowski line element unchanged

Minkowski: Natural set of modes connected to (t, x, y, z) ← associated with the Poincaré group

∂_t: Killing vector of Mink., $\perp$ $t=0$ hypersurf

↳ $\partial_t u(\mathbf{x}) = -i\omega u(\mathbf{x})$ modes are eigenfunctions, $\omega > 0$

Curved space: No longer poincaré symmetry., might exist "natural coords." connected to a timelike Killing vector

$\bar{u}_i(x)$, $u_i(x)$ both complete sets

$$\phi(x) = \sum_i \left[a_i u_i(x) + a^\dagger u_i^*(x)\right], \quad |0\rangle, \quad a_j|0\rangle = 0 \ \forall j$$

$$= \sum_j \left[\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)\right], \quad |\bar{0}\rangle, \quad \bar{a}_j|0\rangle = 0 \ \forall j$$

Bogolubov transformation БОГОЛЮБОВ

$$\bar{u}_j = \sum_i \left[\alpha_{ji} u_i + \beta_{ji} u_i^*\right], \quad \bar{u}_j^* = \sum_i \left[\alpha_{ji}^* u_i^* + \beta_{ji}^* u_i\right]$$

$$u_i = \sum_j \left[\alpha_{ji}^* \bar{u}_j - \beta_{ji}\bar{u}_j^*\right], \quad \bar{u}_i = \sum_j \left[\alpha_{ji}\bar{u}_j^* - \beta_{ji}^*\bar{u}_j\right]$$

$$\phi(x) = \sum_i \left[a_i u_i(x) + a^\dagger u_i^*(x)\right] = \sum_i \left[a_i\left\{\sum_j(\alpha_{ji}^*\bar{u}_j - \beta_{ji}\bar{u}_j^*)\right\} + a^\dagger\left\{\sum_i(\alpha_{ji}\bar{u}_i^* - \beta_{ji}^*\bar{u}_i)\right\}\right]$$

$$= \sum_{ij}\left[a_i\alpha_{ji}^*\bar{u}_j - a_i\beta_{ji}\bar{u}_j^* + a_i^\dagger\alpha_{ji}u_i^* - a^\dagger\beta_{ji}^*\bar{u}_j\right]$$

$$= \sum_{ij}\left[(a_i\alpha_{ji}^* - a^\dagger\beta_{ji}^*)\bar{u}_j + (-a_i\beta_{ji} + a_i^\dagger\alpha_{ji})\bar{u}_j^*\right]$$

$$= \sum_j\left[\sum_i(a_i\alpha_{ji}^* - a_i^\dagger\beta_{ji}^*)\bar{u}_j + \sum_i(-a_i\beta_{ji} + a_i^\dagger\alpha_{ji})\bar{u}_j^*\right] = \sum_j\left(\bar{a}_j\bar{u}_j + \bar{a}_j^\dagger\bar{u}_j^*\right)$$

$$\rightarrow \boxed{\begin{aligned}\bar{a}_j &= \sum_i(a_i\alpha_{ji}^* - a_i^\dagger\beta_{ji}^*)\\ \bar{a}_j^\dagger &= \sum_i(-a_i\beta_{ji} + a_i^\dagger\alpha_{ji})\end{aligned}} \qquad \begin{aligned}a_i &= \sum_j(\bar{a}_j\alpha_{ji} + \bar{a}_j^\dagger\beta_{ji}^*)\\ a_i^\dagger &= \sum_j(\bar{a}_j^\dagger\alpha_{ji}^* + \bar{a}_j\beta_{ji})\end{aligned}$$

A1

$$(u_i, u_j) = \delta_{ij} \quad , \quad (u_i^*, u_j^*) = -\delta_{ij} \quad , \quad (u_i, u_j^*) = 0$$

$$(\phi_1, \phi_2) = -i \int_\Sigma \phi_1 \overleftrightarrow{\partial}_\mu \phi_2^* \sqrt{-g_\Sigma}\, d\Sigma^\mu$$

$$(\bar{u}_i, u_j) = -i \int_\Sigma \bar{u}_i \overleftrightarrow{\partial}_\mu \sum_k (\alpha_{jk} u_k + \beta_{jk} u_k^*) \sqrt{-g_\Sigma}\, d\Sigma^\mu$$

$$= \sum_k \alpha_{jk} (\bar{u}_i, u_k) + \sum_k \beta_{jk} (\bar{u}_i, u$$

$$= \left(\sum_k (\alpha_{ik} u_k + \beta_{ik} u_k^*), u_j \right) = \sum_k \alpha_{ik} \overbrace{(u_k, u_j)}^{\delta_{kj}} + \beta_{ik} \cancelto{0}{(u_k^*, u_j)}$$

$$= \alpha_{ij}$$

$$-(\bar{u}_i, u_j^*) = \left(\sum_k (\alpha_{ik} u_k + \beta_{ik} u_k^*), u_j^* \right) = \sum_k \alpha_{ik} \overbrace{(u_k, u_j^*)}^{0} + \beta_{ik} \overbrace{(u_k^*, u_j^*)}^{-\delta_{kj}}$$

$$= \beta_{ij}$$

$$\boxed{\begin{aligned} (\bar{u}_i, u_j) &= \alpha_{ij} \\ -(\bar{u}_i, u_j^*) &= \beta_{ij} \end{aligned}}$$

$$a_i |0\rangle = 0$$

$$\bar{a}_j |0\rangle = \sum_i \left(\cancelto{0}{+a_i \alpha_{ji}} - a_i^\dagger \beta_{ji}^* \right) |0\rangle = -\sum_i \beta_{ji} |1_i\rangle \neq 0$$

$$a_i |\bar{0}\rangle = \sum_j \left(\bar{a}_j \alpha_{ji} + a_j^\dagger \beta_{ji}^* \right) |\bar{0}\rangle = \sum_j \beta_{ji} |\bar{1}_j\rangle \neq 0$$

$$\boxed{\begin{aligned} \bar{a}_j |0\rangle &= -\sum_i \beta_{ji} |1_i\rangle \neq 0 \\ a_i |\bar{0}\rangle &= \sum_j \beta_{ji} |\bar{1}_j\rangle \neq 0 \end{aligned}}$$

$$N_i = a_i^\dagger a_i$$

$$\langle \bar{0} | N_i | \bar{0} \rangle = \langle \bar{0} | \sum_k (\bar{a}_k^\dagger \alpha_{ki}^* + a_k \beta_{ki}) \sum_l (\bar{a}_l \alpha_{li} + a_l^\dagger \beta_{li}^*) | \bar{0} \rangle$$

$$= \sum_{kl} \langle 1_k | \beta_{ki} \beta_{li}^* | 1_l \rangle = \sum_k |\beta_{ki}|^2$$

$$\boxed{\langle \bar{0} | N_i | \bar{0} \rangle = \sum_l |\beta_{li}|^2}$$

The vacuum of the $\bar{u}_j$ modes consists of $\sum_j |\beta_{ji}|^2$ particles in the u_i modes

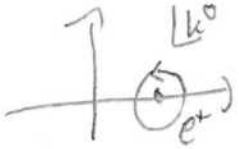
A2

Aside: Pos. freq. Wightman function

$$\langle 0|\phi(x)\phi(x')|0\rangle\,, \quad \phi(x)=\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2E_k}}\left[a_k e^{ikx}+a_k^\dagger e^{-ikx}\right] \text{ with } k^0=E_k=\sqrt{\vec{k}^2+m^2}$$

$$=\langle 0|\int\frac{d^3k}{\sqrt{(2\pi)^3}}\frac{d^3k'}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2E_k 2E_{k'}}}a_{\vec{k}}e^{ikx}a^\dagger_{\vec{k}'}e^{-ik'x'}|0\rangle=\int\frac{d^3k}{(2\pi)^3}d^3k'\frac{1}{\sqrt{4E_kE_{k'}}}e^{ikx-ik'x'}\delta^{(3)}(\vec{k}-\vec{k}')$$

$$=\int\frac{d^3k}{(2\pi)^3}\underbrace{\frac{e^{iE_k(t-t')}}{2E_k}}e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}$$



$$\frac{1}{2\pi i}\int_{\mathcal{C}^+}dk^0\frac{e^{ik^0(t-t')}}{(k^0)^2-E_k^2}=\frac{1}{2\pi i}2\pi i\,\mathrm{Res}_{k^0\to E_k}\frac{e^{ik^0(t-t')}}{(k^0)^2-E_k^2}=\lim_{k^0\to E_k}\frac{e^{ik^0(t-t')}(k^0-E_k)}{(k^0-E_k)(k^0+E_k)}$$

$$k^2=(k^0)^2-\vec{k}^2\,,\ E_k^2=\vec{k}^2+m^2 \qquad =\frac{1}{2E_k}e^{iE_k(t-t')} \quad ✓$$

$$=\frac{1}{i}\int_{\mathcal{C}^+}\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}}{(k^0)^2-E_k^2}=\frac{1}{i}\int_{\mathcal{C}^+}\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}}{k^2-m^2}$$

with k^0 indep. of $\vec{k}$ (not on shell, the pole at $k^2=m^2$ sets $k^0=E_k$ on shell)

trade on shell $\leftrightarrow$ pole

For massless particle: $m=0$

$$G^+(x-x')=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{e^{iE_k(t-t')}}{2E_k}e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}=\int\frac{d^3\vec{k}}{2(2\pi)^3}\frac{e^{i\sqrt{(\vec{k}^2+m^2)}(t-t')}}{\sqrt{\vec{k}^2+m^2}}e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$\underset{|\vec{k}|=E_k}{\overset{m=0}{=}}\int\frac{d^3\vec{k}}{2(2\pi)^3}\frac{e^{i|\vec{k}|(t-t')}e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}}{|\vec{k}|}$$

set coordinate system such that $\vec{k}\parallel\vec{e}_z \Rightarrow \vec{k}\cdot(\vec{x}-\vec{x}')=k|\vec{x}-\vec{x}'|\cos\theta$ (polar coords)

$$=\frac{1}{2}\int_0^{2\pi}d\phi\int_0^\infty\frac{dk}{(2\pi)^3}\int_{-1}^{1}d\cos\theta\,\frac{k^2}{k}e^{ik(t-t')-ik|\vec{x}-\vec{x}'|\cos\theta}=\frac{\pi}{(2\pi)^3}\int_0^\infty dk\,k\underbrace{\left[\frac{e^{-ik|\vec{x}-\vec{x}'|z}}{-ik|\vec{x}-\vec{x}'|z}\right]_{z=-1}^{1}}_{+\frac{2\sin(k|\vec{x}-\vec{x}'|)}{k|\vec{x}-\vec{x}'|}}e^{ik(t-t')}$$

$$=\frac{2\pi}{(2\pi)^3}\int_0^\infty dk\,\frac{\sin(k|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}e^{ik(t-t')}=\frac{1}{(2\pi)^2}\frac{1}{|x-x'|}\int_0^\infty dk\,\sin(k|\vec{x}-\vec{x}'|)\exp(ik(t-t'))\ \Big|\ t\to t-i\varepsilon$$

Mathematica: $\int_0^\infty dk\,\sin(kx)e^{ikt-\varepsilon k}=\frac{x}{(\varepsilon-it)^2+x^2}$

$$=\frac{1}{(2\pi)^2}\frac{1}{\cancel{|x-x'|}}\frac{\cancel{|x-x'|}}{(t-t'-i\varepsilon)^2-|x-x'|^2}=\text{eq. (3.59) Birrell Davies}$$

Unruh Effect (Carroll)

$$ds^2 = -dt^2 + dx^2$$

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau)$$

$$x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau) \quad \rightarrow \quad x^2(\tau) = t^2(\tau) + \frac{1}{\alpha^2}$$

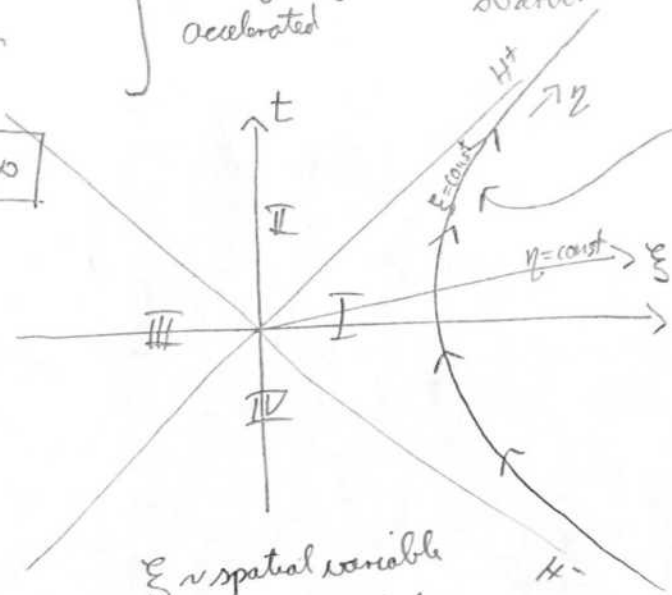
mistake in book? It says α^2

$$a^\mu = \frac{d^2x^\mu}{d\tau^2}, \quad a_\mu a^\mu = \alpha^2 \text{ proper acceleration}$$

observer that is uniformly accelerated

actually $a^\mu = \frac{Dx^\mu}{d\tau^2}$, but flat space $\rightarrow D = d$

want to describe a portion of spacetime with coords of accel. observer



Coordinates (η, ξ), $\boxed{-\infty < \eta, \xi < \infty}$

$$t = \frac{1}{a} e^{a\xi}\sinh(a\eta)$$
$$x = \frac{1}{a} e^{a\xi}\cosh(a\eta)$$

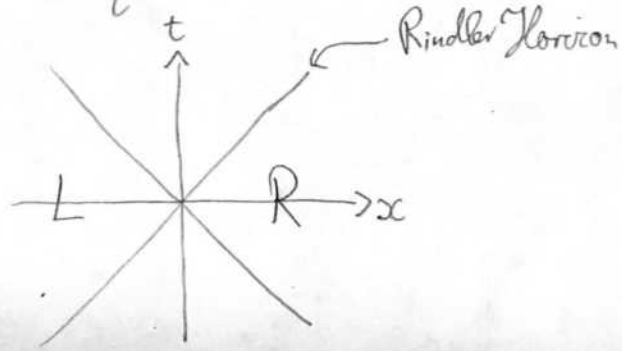
can only represent $x > |t|$

$$x^2 - t^2 = \underbrace{\frac{1}{a^2}e^{2a\xi}}_{\hat{=}\frac{1}{\alpha^2}} \Rightarrow \alpha = \frac{1}{a}e^{a\xi}$$

ξ = const.: observer with $\alpha = \frac{1}{a}e^{a\xi} = \text{const}$

ξ ~ spatial variable

η ~ time coordinate



Right Rindler Wedge R

$$x^2 = \frac{1}{a^2}e^{2a\xi} + t^2 \geq t^2 \Rightarrow x^2 \geq t^2$$

and $x \geq 0$ $\quad x > |t|$

Left Rindler Wedge L

$t = -\dots$ (actually formally time reversed, but that is just coordinates)

$x = -\dots$

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) \quad \text{Rindler Space}$$

↳ indep. of $\eta \Rightarrow \partial_\eta$ Killing vector

$$\partial_\eta = \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x$$

$$= ax\partial_t + at\partial_x \quad \text{boost in } x$$

H^+, H^-: Killing Horizon for ∂_η

Rindler observer: ξ = const

constant acceleration

- Why $\frac{1}{a}e^{a\xi}$? → So the metric is diagonal
- Why a and ξ?
  → Can choose time, acceleration and space at one point.

surface gravity at

AG

$\partial_\eta = ax\partial_t + at\partial_x$, $d\eta = \dots$, $v^\mu = \begin{pmatrix} ax \\ at \\ 0 \\ 0 \end{pmatrix}$, $v_\mu = g_{\mu\nu}v^\nu = \begin{pmatrix} -ax \\ at \end{pmatrix}$

$V = \|\partial_\eta\| = e^{a\xi}$

$v^\mu v_\mu = a^2\underbrace{(t^2-x^2)}_{-\frac{e^{2a\xi}}{a^2}} = -e^{2a\xi}$

$\kappa = \sqrt{\nabla_\mu V \nabla^\mu V}$

$\nabla_\mu V$: $\left.\begin{array}{l} \mu=\xi:\ \nabla_\xi V = ae^{a\xi} \\ \mu=\eta:\ \nabla_\eta V = 0 \end{array}\right\}$ $\begin{pmatrix} 0 \\ ae^{a\xi} \end{pmatrix}$

$\kappa = \sqrt{\nabla_\mu V \nabla^\nu V} = \sqrt{+a^2} = a$

$g_{\mu\nu} = e^{2a\xi}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \to g^{\mu\nu} = e^{-2a\xi}(\dots)$, $\nabla^\mu V = \begin{pmatrix} 0 \\ ae^{-2a\xi} \end{pmatrix}$

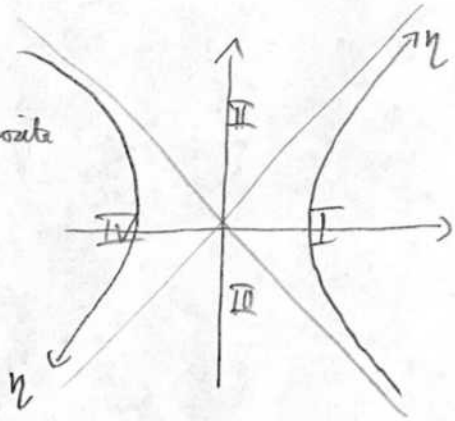
$\boxed{\kappa = a}$

<u>Region IV</u> :

$t = -\frac{1}{a}e^{a\xi}\sinh(a\eta)$

$x = -\frac{1}{a}e^{a\xi}\cosh(a\eta)$ $\quad x<|t|$

$\partial_\eta, \partial_t$ point in opposite directions in IV



Build set of pos/neg. freq. modes wrt. ∂_η

$$\Box\phi = -\partial_t^2 + \partial_x^2 = e^{-2a\xi}\underbrace{\left[-\partial_\eta^2 + \partial_\xi^2\right]\phi(\eta,\xi)} = 0$$

$g_k(\eta,\xi) = \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega\eta + ik\xi}$, $\omega = |k|$ solution

$\partial_\eta g_k(\eta,\xi) = -i\omega g_k$ "positive frequency"

The frequency is positive in the sense that there is a minus sign.

In region IV ∂_η is past directed $\to$ use $\partial_{(-\eta)} = -\partial_\eta$

$$\begin{cases} g_k^{(1)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega\eta+ik\xi} & \text{I} \\ 0 & \text{IV} \end{cases} & \text{pos. modes wrt. } \partial_\eta \text{ in I} \\ g_k^{(2)} = \begin{cases} 0 & \text{I} \\ \frac{1}{\sqrt{4\pi\omega}}e^{+i\omega\eta+ik\xi} & \text{IV} \end{cases} & \text{pos. modes wrt } \partial_{(-\eta)} \text{ in IV} \end{cases} \quad \omega = |k|$$

Can be analytically continued to II, III, but I think we don't care

$$\phi(\eta,\xi) = \int dk\left[\overset{\text{annihilation}}{\hat{b}_k^{(1)}} g_k^{(1)} + \overset{\text{creation}}{\hat{b}_k^{(1)\dagger}} g_k^{(1)*} + (1\to 2)\right] \quad \text{Rindler Space}$$

$$\phi = \int dk\left(\hat{a}_k f_k + \hat{a}_k^\dagger f_k^*\right) \quad \text{Minkowski 2D}$$

A5

$$(\phi_1,\phi_2) = -i\int_\Sigma \left(\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1\right) n^\mu \sqrt{\gamma}\, d^{n-1}x$$

Σ: $\eta = 0$ surface $\Rightarrow n^\mu = \begin{pmatrix}1\\0\end{pmatrix}$ is $\perp \Sigma$

normalization $g_{\mu\nu} n^\mu n^\nu = -e^{2a\xi}(n^0)^2 \overset{!}{=} -1 \Rightarrow n^0 = e^{-a\xi}$

$$\Big\} \; n^\mu = \begin{pmatrix} e^{-a\xi} \\ 0 \end{pmatrix}$$

metric on Σ:

$$ds^2 = e^{+2a\xi}\left[d\xi^2 - d\eta^2\right] = e^{+2a\xi} d\xi$$

$$\Rightarrow \sqrt{\det \gamma} = e^{a\xi}$$

$$(\phi_1,\phi_2) = -i\int_\Sigma \left[\phi_1 \partial_\eta \phi_2^* - \phi_2^* \partial_\eta \phi_1\right] e^{a\xi}\, d\xi$$

$$g_k^{(1)} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi}$$

$$(g_k^{(1)}, g_q^{(1)}) = -i\int_{-\infty}^{\infty}\left[\frac{1}{4\pi\sqrt{\omega_k\omega_q}} e^{-i\omega_k\eta + ik\xi}\, i\omega_q e^{+i\omega_q\eta + iq\xi} - \frac{1}{4\pi\sqrt{\omega_k\omega_q}} e^{+i\omega_q\eta + iq\xi}\, i\omega_k e^{-i\omega_k\eta + ik\xi}\right]_{\eta=0} d\xi$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\sqrt{\frac{\omega_q}{\omega_k}}\, e^{i(k-q)\xi}\, d\xi + \int_{-\infty}^{\infty}\frac{1}{4\pi}\sqrt{\frac{\omega_k}{\omega_q}}\, e^{+i(k-q)\xi}\, d\xi = \delta(k-q)$$

$$(g_k^{(2)}, g_q^{(2)}) = \delta(k-q)$$

$$(g_k^{(1)}, g_q^{(2)}) = 0 \quad \text{(not even support in respective areas)}$$

<u>Modes</u>

Minkowski	Bogolubov	Rindler
f_k, a_k	$\longleftrightarrow$	$g_k^{(1)}, g_k^{(2)}, b_k^{(1)}, b_k^{(2)}$
$a_k\lvert 0_M\rangle = 0$		$b_k^{(1)}\lvert 0_R\rangle = b_k^{(2)}\lvert 0_R\rangle = 0$

Trick for Bogolubov transformation:

Take Rindler Modes and extend to whole spacetime

$$e^{-a(\eta-\xi)} = \begin{cases} a(-t+x) & \text{I} \\ a(t-x) & \text{IV} \end{cases}$$

$$\text{I}: \left.\begin{aligned} t &= a^{-1}e^{a\xi}\sinh(a\eta) \\ x &= a^{-1}e^{a\xi}\cosh(a\eta) \end{aligned}\right\} \; -t+x = \frac{1}{a}e^{a\xi}\overbrace{[\cosh - \sinh](a\eta)}^{e^{-a\eta}} = \frac{1}{a}e^{-a(\eta-\xi)}$$

$$\text{IV}: \begin{aligned} t &= - \dots \\ x &= - \dots \end{aligned}$$

$$e^{+a(\eta-\xi)} = \begin{cases} a(t+x) \\ -a(t+x) \end{cases}$$

Look at $g_k^{(1)}$, $k>0$: $(\omega = k)$: $g_k^{(1)} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega(\eta-\xi)} = \frac{1}{\sqrt{4\pi\omega}}\Big[\overbrace{e^{-a(\eta-\xi)}}^{a(-t+x)}\Big]^{i\omega/a} = \frac{1}{\sqrt{4\pi\omega}}\, a^{i\frac{\omega}{a}}(-t+x)^{i\frac{\omega}{a}}$

↳ just use this new expression in all of spacetime

Same for $g_k^{(2)}$, $k>0$, $\omega = k$: $g_k^{(2)} = \frac{1}{\sqrt{4\pi\omega}}\, a^{-i\omega/a}(-t-x)^{-i\omega/a}$

A6

$g_k^{(1)}, k>0$: $\quad g_k^{(1)} = \frac{1}{\sqrt{4\pi\omega}}\, a^{i\frac{\omega}{a}} (-t+x)^{i\frac{\omega}{a}}$

$g_k^{(2)}, k>0$: $\quad g_k^{(2)} = \frac{1}{\sqrt{4\pi\omega}}\, a^{-i\frac{\omega}{a}} (-t-x)^{-i\omega/a}$

$$g_{-k}^{(2)*} = \frac{1}{\sqrt{4\pi\omega}} \left[e^{i\omega\eta + ik\xi}\right]^*_{k\to -k} = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega\eta + ik\xi} = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega(\eta-\xi)} = \frac{a^{i\omega/a}}{\sqrt{4\pi\omega}} (t-x)^{i\omega/a} = \frac{a^{i\omega/a}}{\sqrt{4\pi\omega}} \left[\overset{e^{-i\pi}}{(-1)}(-t+x)\right]^{i\omega/a}$$

$$= \frac{a^{i\omega/a}}{\sqrt{4\pi\omega}}\, e^{-i^2\omega/a} (-t+x)^{i\omega/a} = \frac{e^{\omega/a}}{\sqrt{4\pi\omega}}\, a^{i\omega/a} (-t+x)^{i\omega/a}$$

$\hookrightarrow e^{-\omega/a} g_{-k}^{(2)*} = \frac{1}{\sqrt{4\pi\omega}}\, a^{i\omega/a} (-t-x)^{i\omega/a}$ same as $g_k^{(1)}$, but $g^{(1)}$ is only def. for I and $g^{(2)}$ for IV

Thus: $\underbrace{\left[g_k^{(1)}\right.}_{\mathrm{I}} + \underbrace{\left.e^{-\pi\omega/a} g_{-k}^{(2)*}\right]}_{\mathrm{IV}} = \frac{1}{\sqrt{4\pi\omega}}\, a^{i\omega/a} (-t+x)^{i\omega/a}$ defined for all of Σ $(t=0)$

for $k<0$ one gets the same!

$$h_k^{(1)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left[e^{\pi\omega/2a} g_k^{(1)} + e^{-\pi\omega/2a} g_{-k}^{(2)*}\right]$$ normalized — analytic extension of $g^{(1)}$ to IV

$$h^{(2)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left[e^{\pi\omega/2a} g_k^{(2)} + e^{-\pi\omega/2a} g_{-k}^{(1)*}\right]$$ analytic extension of $g^{(2)}$ to I

$$\left(h_k^{(1)}, h_q^{(2)}\right) = \delta(k-q)$$

$$\phi = \int dk \left[c_k^{(1)} h_k^{(1)} + c_k^{(1)\dagger} h_k^{(2)*} + c_k^{(2)} h_k^{(2)} + c_k^{(2)\dagger} h_k^{(2)*} \right]$$

Reminder: Bogolubov $\phi = \sum_i \left[a_i f_i + a_i^\dagger f_i^*\right]$
$= \sum_i \left[b_i g_i + b_i^\dagger g_i^*\right]$

$$g_i = \sum_j \left(\alpha_{ij} f_i + \beta_{ij} f_i^*\right)$$
$$f_i = \sum_j \left(\alpha_{ji}^* g_j - \beta_{ij} g_j^*\right)$$

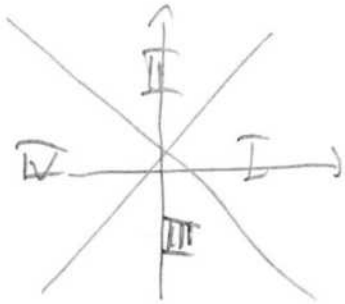
$\downarrow$ equivalently

$$a_i = \sum_j \left(\alpha_{ji} b_j + \beta_{ji}^* b_j^\dagger\right)$$
$$b_i = \sum_j \left(\alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger\right)$$

A7

$$\phi = \int dk \left[a_k f_k + a_k^\dagger f_k^* \right] \qquad \text{Minkowski}$$

$$\phi = \int dk \Big[\underbrace{b_k^{(1)} g_k^{(1)} + b_k^{(1)\dagger} g_k^{(1)*}}_{\text{only support on I}} + \underbrace{b_k^{(2)} g_k^{(2)} + b_k^{(2)\dagger} g_k^{(2)*}}_{\text{only on IV}} \Big] \qquad \text{Rindler}$$



$$h_k^{(1)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left[e^{\pi\omega/2a} g_k^{(1)} + e^{-\pi\omega/2a} g_{-k}^{(2)*} \right]$$

extension of $g_k^{(1)}$ to whole $t=0$ surface

$$h_k^{(2)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left[e^{\pi\omega/2a} g_k^{(2)} + e^{-\pi\omega/2a} g_{-k}^{(1)} \right]$$

$$\phi = \int dk \left[c_k^{(1)} h_k^{(1)} + c_k^{(1)\dagger} h_k^{(1)*} + c_k^{(2)} h_k^{(2)} + c_k^{(2)\dagger} h_k^{(2)*} \right]$$

Basically Bogolubov transform. from $g_k^{(1,2)} \to h_k^{(1,2)}$

Corresponding transformation for $b_k \to c_k$: just replace $h \to b$

$g \to c$

$* \to \dagger$

$$b_k^{(1)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left[e^{\pi\omega/2a} c_k^{(1)} + e^{-\pi\omega/2a} c_{-k}^{(2)\dagger} \right]$$

$$b_k^{(2)} = \ldots$$

$$n_R^{(1)}(k) = b_k^{(1)\dagger} b_k^{(1)} \longrightarrow c_k$$

WHY?

c_k and Minkowski must share the vacuum state (explanation complicated, $\boxed{\text{No Do}}$)

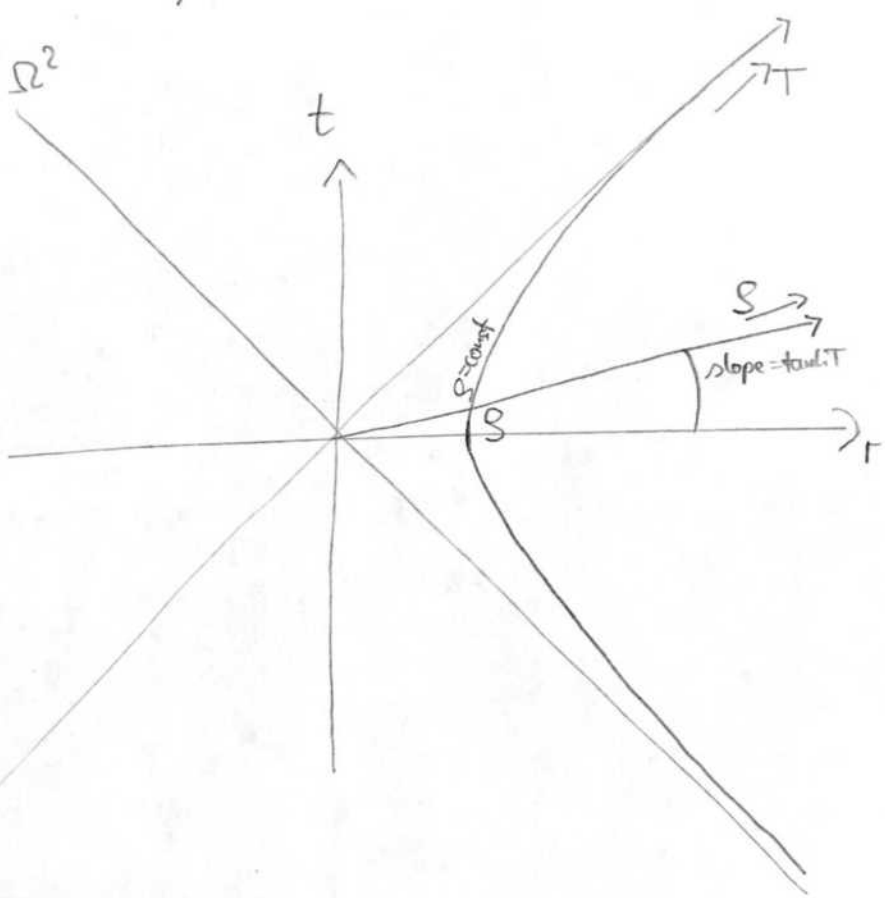
$$\langle 0_M | b_k^{(1)} b_k^{(1)} | 0_M \rangle = \frac{1}{e^{2\pi\omega/a} - 1} \delta(0) \qquad , \; T = \frac{a}{2\pi}$$

A8

Rindler Space in ρ, T coordinates

$$t = \rho \sinh T \quad , \quad dt = \sinh T \, d\rho + \rho \cosh T \, dT$$
$$r = \rho \cosh T \quad , \quad dr = \cosh T \, d\rho + \rho \sinh T \, dT$$

$$g = ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$
$$= -(\sinh T \, d\rho + \rho \cosh T \, dT)^2 + (\cosh T \, d\rho + \rho \sinh T \, dT)$$
$$= (-\sinh^2 T + \cosh^2 T) d\rho + \rho^2 (-\cosh^2 T + \sinh^2 T) dT^2 + \overbrace{(-\sinh \cosh + \cosh \sinh)}^{0} dT d\rho + \%$$
$$g = -\rho^2 dT^2 + d\rho^2 + \rho^2 \cosh^2 T \, d\Omega^2$$

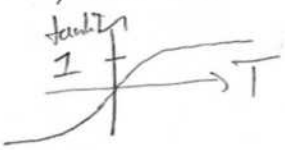


Constant ρ

$$r^2 - t^2 = \rho^2 \Rightarrow t(r) = \pm\sqrt{r^2 - \rho^2}$$
$$\boxed{t = 0 \Rightarrow r = \rho}$$

Constant T

$$\frac{t}{r} = \tanh T \quad , \quad t(r) = r \tanh T$$



$T \to \infty$: slope $\to 1$

$T \to -\infty$: slope $\to -1$

$T = 0$: slope $= 0$

Observer at $\rho = \text{const.} \to V = V^0 \partial_T \to$ sees T as his time

$$t = \rho \sinh T$$
$$r = \rho \cosh T$$

proper time $d\tau = \sqrt{+\rho^2 dT + \underbrace{d\rho^2}_{0} + \rho^2 \cosh^2 T \underbrace{d\Omega^2}_{0}} = \rho \, dT$

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} = \frac{d^2 x^\mu}{\rho^2 dT^2} = \begin{pmatrix} \frac{1}{\rho} \sinh T \\ \frac{1}{\rho} \cosh T \end{pmatrix} \quad , \quad a^\mu a_\mu = \frac{1}{\rho^2}(\cosh^2 - \sinh^2) = \frac{1}{\rho^2}$$

$\boxed{a = \frac{1}{\rho}}$ proper accel. of observer at $\rho = \text{const.}$

$$\Rightarrow T = \frac{a}{2\pi} = \frac{1}{2\pi\rho}$$