# HEP Seminar 2019: CFT

## Outline:

Ref: Polchinski, 1511.04074
Vol. I., Complex variables

Today we hope to cover,

- Primary fields & Radial Quantization (conserved currents, charges)
- Stress energy tensor & OPE's (Radial ordering to OPE)
- ~~Algebras~~ Highest Weight states / Unitarity Bounds.
- Ward ID's

**Notation** $\delta_{m,n} \Rightarrow m=n \Rightarrow m-n=0 \Rightarrow \delta_{m-n,0} \equiv \delta_{m-n}$

- Primary fields & Radial Quantization

Definitions: Consider a field $\phi(z,\bar{z})$

if $\phi$ is harmonic $\partial\bar{\partial}\phi = 0$

Then $\phi(z,\bar{z}) = \underbrace{\phi(z)}_{\text{holomorphic or chiral}} + \underbrace{\bar{\phi}(\bar{z})}_{\text{anti-holomorphic or anti-chiral}}$

it is convient to take a basis of local operators which are eigenstates of the Ridgid conformal trans $z \to z' = \lambda z$

Notation of 1511.04074

Then $$A'(z',\bar{z}') = \lambda^{-h}\bar{\lambda}^{-\tilde{h}} A(z,\bar{z})$$ $\boxed{\phi'(z,\bar{z}) = \lambda^{h}\bar{\lambda}^{\tilde{h}}\phi(\lambda z,\bar{\lambda}\bar{z})}$

Under a General conformal transformation

$$A'(z',\bar{z}') = (\partial_z z')^{-h}(\partial_{\bar{z}}\bar{z}')^{-\tilde{h}} A(z,\bar{z}) \quad *$$

$(h, \tilde{h})$ are known as weights.

- $h + \tilde{h}$ is the dimension of $\phi$, (related to scaling)
- $h - \hat{h}$ is the spin (related to rotations)

Primary fields: transform as $*$
$\forall$ conformal trans.

quasi-primary : if $\phi$ transforms as $*$ for only Global conformal trans it is quasi-primary.

"a Primary field is also a quasi-Primary. But a quasi-primary is not nesecary to be a primary. A Primary has an infinite set of quasi-primary desecendents"

Under $z \to z' = f(z) = z + \epsilon(z)$

$$\left(\frac{\partial f}{\partial z}\right)^{+h} = (1 + \partial\epsilon)^{+h} = 1 + h\partial\epsilon + O(\epsilon^2)$$

$$\Phi(z + \epsilon(z), \bar{z}) = \Phi(z) + \epsilon(z)\partial\phi + \cdots$$

Therefore

with $\phi = (\partial f)^{-h}(\bar{\partial} f)^{-\hat{h}}\phi$

$$\delta\phi = \phi' - \phi = (\partial f)^{+h}(\bar{\partial} f)^{+\hat{h}}\phi - \phi$$

$$= (1 + h\partial\epsilon)(1 + \hat{h}\bar{\partial}\bar{\epsilon})(\phi + \epsilon\partial\phi + \bar{\epsilon}\bar{\partial}\phi) - \phi$$

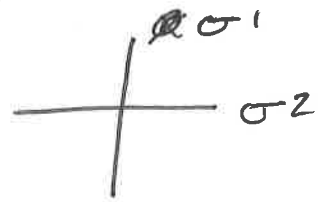
infinitesimal transformation of a Primary field

$$\boxed{\delta\phi = (\epsilon\partial + h\partial\epsilon + \bar{\epsilon}\bar{\partial} + \hat{h}\bar{\partial}\bar{\epsilon})\phi}$$

Now lets continue our 2d investigations

We take $(\sigma^1, \sigma^2) \sim (t, x)$

$(\sigma^0, \sigma^1) \sim (t, x)$

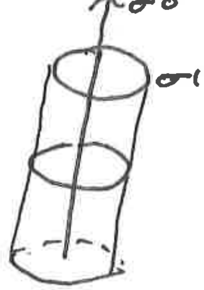



as coordinates

Consider compactifying $\sigma^1$ on a circle of Radius $R=1$.

This Theory lives on an infinite Cylinder

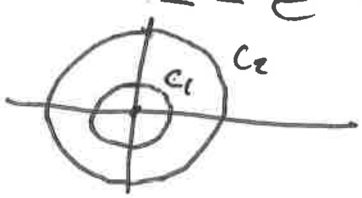
$\sigma^1 \sim \sigma^1 + 2\pi$




easily accomplished with $\sigma^0 + i\sigma^1 = w$

We can mapp to The Complex Plane via

$z = e^{w} = e^{\sigma^0 + i\sigma^1} = r e^{i\sigma^1}$




- infinite past

$c_1$ has $t = \sigma_1^0 < \sigma_2^0$

time translation $\sigma^0 \to \sigma^0 + a$ $e^{a} e^{w}$ (dilation)

Space translation $\sigma^1 \to \sigma^1 + a$ $e^{ia} e^{w}$ (rotation)

recall $l_0 = -\frac{1}{2} r\partial_r + \frac{i}{2}\partial_\theta$ $\quad \bar{l}_0 = -\frac{1}{2} r\partial_r - \frac{i}{2}\partial_\theta$

$\therefore$ $l_0 + \bar{l}_0 = -r\partial_r$ $\quad$ ; $\quad$ $i(l_0 - \bar{l}_0) = -\partial_\theta$

(Think $l_m = -z^{m+1}\partial$)

Therefore, Recall from QM The Hamiltonian is The generator for time translation, we have

$$H = L_0 + \bar{L}_0 \ , \quad P = i(L_0 - \bar{L}_0)$$

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~~Now~~ in order to define The Path integral we will need to define initial states in The $w$ coor. This corresponds to the bottom of The cylinder. In the $z = e^w$ coor. This corresponds to The Point at The origin

This defines a local op at The origin. Known as The vertex op. associated with The state.

let $\quad \phi(z,\bar{z}) = \sum_{n,m} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n} \qquad -m-h \leq 0$

The $\quad |\phi\rangle_{in} = \lim_{z,\bar{z} \to 0} \phi(z,\bar{z})|0\rangle \ , \quad$ with $\boxed{\begin{matrix} -h \leq m \\ -\tilde{h} \leq n \end{matrix}}$

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The stress energy tensor & OPE's

Recall for $\sigma^\mu \to \sigma^\mu + \epsilon^\mu(x)$ we found $T_{\mu\nu}\epsilon^\mu$ as our current and found it to be conserved

$$\partial^\mu T_{\mu\nu} = 0$$

Consider

$$\delta S = -\frac{1}{2}\int d^2\sigma\, T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)$$

Now Recall $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \delta_{\mu\nu}\,\partial\cdot\epsilon$

$$= -\frac{1}{2}\int d^2\sigma\, T^{\mu\nu}(\delta_{\mu\nu}\,\partial\cdot\epsilon)$$

$$= -\frac{1}{2}\int d^2\sigma\, T^{\mu}{}_{\mu}\,\partial\cdot\epsilon = 0$$

implies $\boxed{T^{\mu}{}_{\mu} = 0}$

The stres-energy tenser in a conformal Theory is traceless.

$$T_{\mu\nu} \to \frac{\partial x^\rho}{\partial x^\mu}\frac{\partial x^\sigma}{\partial x^\nu}T_{\rho\sigma}$$

$\boxed{\text{use } z \text{ instead.}}$

$w = \sigma^1 + i\sigma^2$

$\Rightarrow \sigma^1 = \frac{1}{2}(w + \bar{w})$

$\sigma^2 = \frac{-i}{2}(w - \bar{w})$

$$\Rightarrow \frac{\partial x^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -i/2 & i/2 \end{pmatrix}$$

$\sigma^1, \sigma^2 \Rightarrow z = e^{\sigma^1}e^{i\sigma^2} = z_1 + iz_2$

$\bar{z} = e^{\sigma^1}e^{-i\sigma^2} = z_1 - iz_2$

$z\bar{z} = e^{2\sigma^1}$

$\frac{1}{2}\ln|z|^2 = \sigma^1$

$-\frac{i}{2}\ln z/\bar{z} = \sigma^2$

$T^\mu{}_\mu = T^1{}_1 + T^2{}_2 = 0$

$T_{11} = -T_{22}$

$$\Rightarrow T_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}T_{\alpha\beta}$$

$$\Rightarrow T_{11} = \frac{\partial x^\alpha}{\partial x^1}\frac{\partial x^\beta}{\partial x^1} = \frac{1}{4}(T_{11} - T_{22} - 2iT_{12}) = \frac{1}{2}(T_{11} - iT_{12})$$

$$T_{22} = \frac{\partial x^\alpha}{\partial x^2}\frac{\partial x^\beta}{\partial x^2} = \frac{1}{4}(T_{11} - T_{22} + 2iT_{12}) = \frac{1}{2}(T_{11} + iT_{12})$$

$$T_{12} = \frac{\partial x^\alpha}{\partial x^1}\frac{\partial x^\beta}{\partial x^2} = i\frac{1}{4}(T_{11} + T_{22} - i\cancel{T_{21}} + i\cancel{T_{12}}) = 0$$

$T_{21} = 0$

now we have ↙ its Euclidean

$$\partial^\mu T_{\mu\nu} = \partial_\mu T_{\mu\nu} = 0$$

What did we use?
1.) traceless.
2.) conserved
3.) complex coordinates

$$\partial_1 T_{1\nu} + \partial_2 T_{2\nu} = 0$$

$$\rightarrow \quad \partial_\mu T_{\mu\nu} \rightarrow \boxed{\begin{array}{l} \partial_1 T_{11} + \partial_2 T_{21} = 0 \\ \partial_1 T_{12} + \partial_2 T_{22} = 0 \end{array}}$$

now

$$T_{zz} \equiv T = \tfrac{1}{2}(T_{11} - iT_{12}) \Rightarrow T_{11} = T + \bar{T}$$

$$T_{\bar{z}\bar{z}} = \bar{T} = \tfrac{1}{2}(T_{11} + iT_{12}) \qquad T_{12} = i(T - \bar{T})$$

$$T_{z\bar{z}} = 0 \qquad T_{11} = -T_{22} \qquad T_{22} = -(T + \bar{T})$$

we also have $\partial_1 = \partial + \bar{\partial}$, $\partial_2 = i(\partial - \bar{\partial})$

$$\Rightarrow \partial_1 T_{11} + \partial_2 T_{21} = (\partial + \bar{\partial})(T + \bar{T}) + i(\partial - \bar{\partial})(i(T - \bar{T})) = 0$$

$$= \partial T + \partial\bar{T} + \bar{\partial}T + \bar{\partial}\bar{T} - [\partial T - \partial\bar{T} - \bar{\partial}T + \bar{\partial}\bar{T}] = 0$$

$$= \partial\bar{T} + \bar{\partial}T = 0$$

$$\partial_1 T_{12} + \partial_2 T_{22} = i(\partial + \bar{\partial})(T - \bar{T}) + i(\partial - \bar{\partial})(-T - \bar{T})$$

$$= \partial T - \partial\bar{T} + \bar{\partial}T - \bar{\partial}\bar{T} - [\partial T + \partial\bar{T} - \bar{\partial}T - \bar{\partial}\bar{T}] = 0$$

$$= -\partial\bar{T} + \bar{\partial}T = 0 \Rightarrow \bar{\partial}T = \partial\bar{T}$$

$$;\quad 2\partial\bar{T} = 0 \quad \text{or} \quad \partial\bar{T} = 0 \rightarrow \bar{T} = \bar{T}(\bar{z})$$

$$2\bar{\partial}T = 0 \qquad \bar{\partial}T = 0 \qquad T = T(z)$$

∴ We have a Chiral & Anti-Chiral Piece of The energy momentum tensor.

Because $J_\mu = T_{\mu\nu}\epsilon^\nu$ is conserved

There is a conserved charge

$$Q = \int dx^1 J_0 \quad \text{or} \quad \int dx^2 J_1$$

and $Q$ is the generator of the symmetry transformations.

$$\delta A = [Q, A]$$

evaluated at equal times.

In radial quantization $\delta A$ is evaluated at constant $|z|$. The space integral becomes a contour integral. $J_\mu = T_{\mu 1}\epsilon^1 + T_{\mu 2}\epsilon^2$

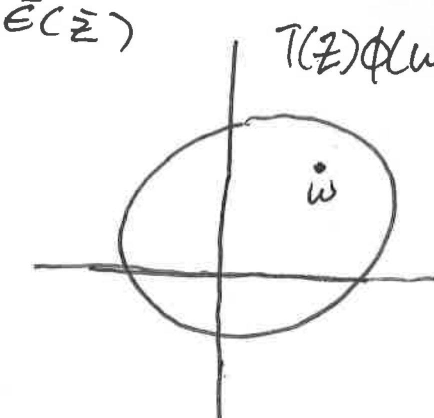
$$\Rightarrow \quad J_1 = T_{11}\epsilon^1 + T_{12}\epsilon^2$$

$$= (T+\bar{T})\epsilon^1 + i(T-\bar{T})\epsilon^2$$

$$= T(\epsilon^1 + i\epsilon^2) + \bar{T}(\epsilon^1 - i\epsilon^2)$$

$$J_1 = T(z)\epsilon(z) + \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$$

$$\Rightarrow \quad Q = \frac{1}{2\pi i}\oint_C (dz\, T\epsilon + d\bar{z}\, \bar{T}\bar{\epsilon})$$



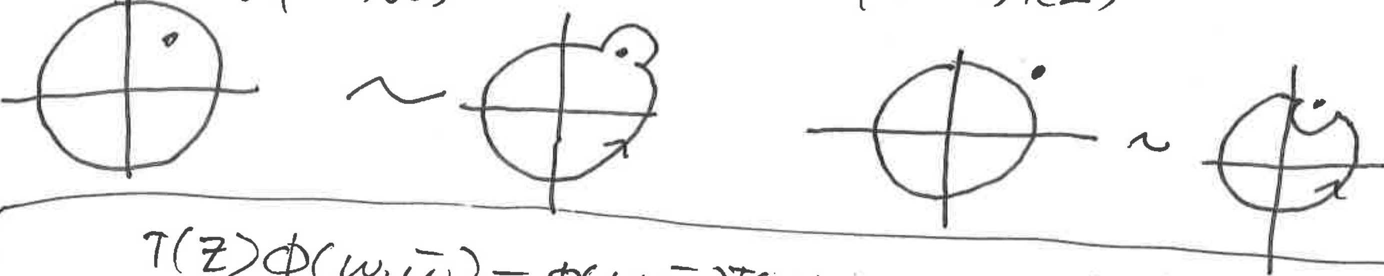

Now consider

$$\delta\phi(w,\bar{w}) = [Q,\phi] = \frac{1}{2\pi i}\left[\int dz\, T\epsilon + d\bar{z}\,\bar{T}\bar{\epsilon},\ \phi(w,\bar{w})\right]$$

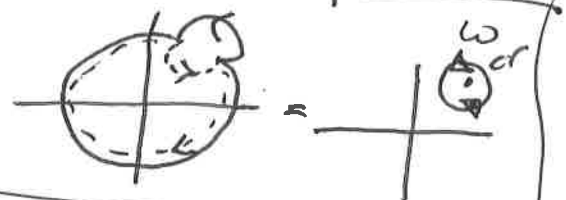
$$= \frac{1}{2\pi i}\int dz\, T\epsilon\,\phi(w,\bar{w}) + d\bar{z}\,\bar{T}\bar{\epsilon}\,\phi(w,\bar{w}) + dz\,\phi(w,\bar{w})\,T\epsilon + \phi(w,\bar{w})\,d\bar{z}\,\bar{T}\bar{\epsilon}$$

Now introduce the radial ordering. This is synonymous with T ordered products in QFT

$T(z)\phi(w,\bar{w})$ $\quad\quad$ $\phi(w,\bar{w})T(z)$



$T(z)\phi(w,\bar{w}) - \phi(w,\bar{w})T(z)$



$$R[A(z),B(w)] = \begin{cases} A(z)B(w) & |w|<|z| \\ B(w)A(z) & |z|<|w| \end{cases}$$

With This in mind we can Rewrite as

$$\delta\phi(w,\bar{w}) = \frac{1}{2\pi i}\left(\oint_{|w|<|z|} - \oint_{|z|<|w|}\right)\left(dz\,\epsilon(z)R[T(z)\phi(w,\bar{w})] + d\bar{z}\,\bar{\epsilon}(\bar{z})R[\bar{T}(\bar{z}),\phi(w,\bar{w})]\right)$$

Expand and check, $d\tilde{z} = \epsilon(z)dz$

$$\frac{1}{2\pi i}\oint_{|w|<|z|} d\tilde{z}\,R[T(z),\phi(w,\bar{w})] + \frac{1}{2\pi i}\oint_{|w|<|z|} d\tilde{\bar{z}}\,R[\bar{T}(\bar{z}),\phi(w,\bar{w})]$$

$$-\frac{1}{2\pi i}\oint_{|z|<|w|} d\tilde{z}\,R[T(z),\phi(w,\bar{w})] - \frac{1}{2\pi i}\oint_{|z|<|w|} d\tilde{\bar{z}}\,R[\bar{T}(\bar{z}),\phi(w,\bar{w})]$$

$$= \frac{1}{2\pi i}\left(-\oint \phi(w,\bar{w})d\tilde{z}\,T(z) + \oint T(z)d\tilde{z}\,\phi(w,\bar{w}) + \oint d\tilde{\bar{z}}\,\bar{T}(\bar{z})\phi(w,\bar{w}) - \oint d\tilde{\bar{z}}\,\phi(w,\bar{w})\bar{T}(\bar{z})\right)$$

$$= \frac{1}{2\pi i}\left[\oint dz\,T\epsilon + d\bar{z}\,\bar{T}\bar{\epsilon},\ \phi(w,\bar{w})\right] \checkmark$$

lets simplify The boxed variation with boxed image, what This does Reduce $\oint_{|w|<|z|} - \oint_{|z|<|w|} = \oint_{c'}$

Where $c'$ is a circle enclosing $w$

$$\delta\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{c'} dz\,\epsilon(z)R[T(z),\phi(w,\bar{w})] + d\bar{z}\,\bar{\epsilon}(\bar{z})R[\bar{T}(\bar{z})\phi(w,\bar{w})]$$

However we know that

$$\delta\Phi = +h\,\partial\epsilon\,\Phi + \epsilon\,\partial\Phi + \bar{h}\,\bar{\partial}\bar{\epsilon}\,\Phi + \bar{\epsilon}\,\bar{\partial}\Phi$$

So we can use Cauchys formula

$$f^{(n)}(a_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)\,dz}{(z-a_0)^{n+1}}$$

Clearly

$$+h\,\partial\epsilon\,\Phi = \frac{1}{2\pi i}\oint_c dz\ \frac{+h\,\epsilon(z)\,\phi(w,\bar{w})}{(z-w)^2}$$

$$\epsilon\,\partial\phi = \frac{1}{2\pi i}\oint_c dz\ \frac{\epsilon(z)\,\partial_w\phi(w,\bar{w})}{(z-w)}$$

looking back we can see

$$\frac{1}{2\pi i}\oint_c dz\ \frac{+h\,\epsilon(z)\,\phi(w,\bar{w})}{(z-w)^2} + \frac{\epsilon(z)\,\partial_w\phi(w,\bar{w})}{(z-w)^1} + \text{Reg} + \text{anti-chiral}$$

$$= \frac{1}{2\pi i}\oint_c dz\,\epsilon\,R[T(z),\phi(w,\bar{w})]$$

$$\Rightarrow \boxed{R[T(z),\phi(w,\bar{w})] = +h\frac{\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\phi(w,\bar{w})}{(z-w)} + \text{Reg}}$$

This is an expression ~~for~~ for the radial ordering in terms of an OPE!

$\hookrightarrow$ Example Consider $\partial\phi(w)$ $\longrightarrow$

$$\delta \partial\phi = +h\,\partial^2\epsilon\,\phi + (1+h)\,\partial\epsilon\,\partial\phi + \epsilon\,\partial^2\phi$$

$\epsilon$

$$\Rightarrow \oint_C R[T(z)\phi(w)]\epsilon(z)\,dz = \oint_C +\frac{2h\,\epsilon(z)\phi(w)}{(z-w)^3} + \frac{(1+h)\,\epsilon\,\partial\phi}{(z-w)^2} + \epsilon\frac{\partial^2\phi}{(z-w)}$$

$$\Rightarrow R[T(z)\phi(w)] = +\frac{2h\,\phi(w)}{(z-w)^3} + \frac{(1+h)\,\partial\phi}{(z-w)^2} + \frac{\partial^2\phi}{(z-w)}.$$

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A natural Question to ask Then is
The stress-Energy tensor a Primary Field?
We define

$$T(z) = \sum_n z^{-n-2} L_n, \quad L_n = \frac{1}{2\pi i}\oint \frac{dz}{z}\, z^{n+2}\, T(z)$$

for $\epsilon(z) = +\epsilon_n z^{n+1}$

$$Q_n = \oint \frac{dz}{2\pi i}\, T(z)\epsilon(z) = +\epsilon_n L_n$$

Put $\epsilon_n = 1$

$$[L_n, L_m] = \left[\oint \frac{dz}{2\pi i} T(z)\epsilon(z), \oint \frac{dw}{2\pi i} T(w)\epsilon(w)\right]$$

$$= \oint \frac{dz\,dw}{(2\pi i)^2}\; T(z)T(w)\,z^{n+1} w^{m+1} - T(w)T(z)\,z^{n+1} w^{m+1}$$

or

$$= \frac{1}{(2\pi i)^2}\oint_C dz\,dw\, R[T(z), T(w)]$$

$$= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n} = (m-n)L_{m+n} + \frac{c\,\delta_{m+n} m^3}{12} - \frac{m c\,\delta_{m+n}}{12}$$

first note that

$$\frac{1}{2\pi i}\oint dz\ z^n = \delta_{n+1}$$

$$\Rightarrow \quad \frac{c}{12}\delta_{n+m}(m^3-m) = \frac{c}{12}\oint\frac{dz}{2\pi i}\ \underbrace{z^{n+m-1}}_{z^{n+1+m-2}}(m^3-m)$$

$$= \frac{c}{12}\oint\frac{dz}{2\pi i}\ z^{n+1}z^{m-2}(m^3-m)$$

$$\lim_{w\to z}\ \frac{c}{2}\oint\frac{dz}{(2\pi i)}\ z^{n+1}\frac{\partial_w^3}{3!}w^{m+1}$$

$$= \frac{c}{2}\oint\frac{dz\,dw}{(2\pi i)^2}\ \frac{z^{n+1}w^{m+1}}{(z-w)^4}$$

So the central charge emerges as a term

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4}$$

in the OPE.

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$$(m-n)L_{m+n} = 2(m+1)L_{m+n} - (m+n+2)L_{m+n}$$

$$= \underbrace{\frac{1}{2\pi i}\oint dz\ z^{m+n+1}T(z)2(m+1)}_{} - \frac{1}{2\pi i}\oint dz\ z^{m+n+2}T(z)\times(m+n+2)$$

$$= \frac{1}{2\pi i}\oint dz\ 2z^{n+1}z^m(m+1)T(z)$$

$$= \frac{1}{2\pi i}\lim_{w\to z}\oint dz\ 2z^{n+1}\partial_w w^{m+1}\ T(z) = \frac{1}{2\pi i}\oint dw\,dz\ 2\frac{z^{n+1}w^{m+1}T(z)}{(z-w)^2}$$

$$-\frac{1}{2\pi i}\oint dz \quad z^{m+n+1}\,T(z)\,(m+2+n)$$

$$= -\frac{1}{2\pi i}\oint dz \quad \partial_z (z^{m+n+2})\,T(z)$$

$$= -\frac{1}{2\pi i}\oint dz \quad \underbrace{\partial_z\left(z^{m+n+2}\,T(z)\right)} - z^{m+n+2}\,\partial_z T(z)$$

write $z = z_0 + e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$\partial_z = \frac{d\theta}{dz}\partial_\theta = -i e^{-i\theta}\partial_\theta$

$\Rightarrow$

$$= -\frac{1}{2\pi i}\oint_0^{2\pi} d\theta \frac{\partial}{\partial\theta}\left((z_0+e^{i\theta})^{m+n+2}\,T(z_0+e^{i\theta})\right)$$

$$= -\frac{1}{2\pi i}\left[(z_0+1)^{m+n+2}\,T(z_0+1) - (z_0+1)^{m+n+2}\,T(z_0+1)\right]$$

$= 0$

Second piece

$$= +\frac{1}{2\pi i}\oint dz \quad z^{m+n+2}\,\partial_z T(z)$$

$$= \frac{1}{2\pi i}\oint dz\, z^{n+1} z^{m+1}\,\partial_z T(z)$$

$$= \frac{1}{2\pi i}\oint dz\,dw\, \frac{z^{n+1} w^{m+1}\,\partial_z T(z)}{(z-w)}$$

all together

$$[L_n, L_m] - \frac{1}{2\pi i}\oint dz\,dw\, z^{n+1} w^{m+1}\left[\frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial_z T(z)}{(z-w)}\right]$$

$$= \frac{1}{2\pi i}\oint dz\,dw\, R[T(z)T(w)]$$

∴ We have found the radial ordering for $T(z)$. by insisting $[L_m, L_n]$ obeyed The Virasoro algebra.

$$R[T(w)T(z)] = \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial_z T(z)}{(w-z)}$$

The radial ordering is expressed in terms of The operator product expansion.

"The product of two operators can be expressed as a string/series of operators at a single Point."

Furthermore we see that $T(z)$ is <u>not</u> in general a Primary field.

Why? Consider

$$[L_m, \phi_n], \text{ with } \phi = \sum z^{-m-h} \phi_m$$

acc

$$\Rightarrow \frac{1}{2\pi i}\oint z^{n+h-1}\phi \, dz = \sum_m \frac{1}{2\pi i} \phi_m \oint dz \, z^{n-m-1}$$

$$= \sum_m \phi_m \delta_{n-m} = \phi_n$$

$$\phi_n = \frac{1}{2\pi i}\oint z^{n+h-1} \phi \, dz$$

$$[L_m, \phi_n] = \frac{1}{2\pi i}\oint dz dw R[T(z), \phi(w)] z^{m+1} w^{n+h-1}$$

$$= \frac{1}{2\pi i}\oint dz dw \, z^{m+1} w^{n+h-1}\left[\frac{h}{(z-w)^2}\phi(w) + \frac{1}{(z-w)}\partial_w \phi(w)\right]$$

$$= \frac{1}{2\pi i} \oint dz\, dw\, z^{m+1} w^{n+h-1} \left[ \frac{h\phi(w)}{(z-w)^2} + \frac{1}{(z-w)} \partial_w \phi(w) \right]$$

$$= \frac{1}{2\pi i} \oint dw\, (m+1) h\, w^{m+n+h-1} \phi(w) + w^{m+n+h} \partial_w \phi(w)$$

$$= \quad h(m+1)\phi_{m+n} + \frac{1}{2\pi i} \oint dw\, \partial_w (w^{q} \phi) - \partial_w w^{q} \phi$$

$$= h(m+1)\phi_{m+h} \quad - (m+n+h)\phi_{m+n}$$

$$= ( hm + \cancel{h} - m - n - \cancel{h} ) \phi_{m+n}$$

$$= ( m(h-1) - n ) \phi_{m+n} \qquad \text{for all } \quad m,n \in \mathbb{Z}$$

if This only holds for $m = 0, \pm 1$ $\phi$ is quasi-primary

$T$ - is not primary it is quasi-primary of dimension $(2,0)$

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## highest wieght states & descendents

Consider $|h\rangle = \phi(0)|0\rangle$ $\phi$ - Chiral $(h,0)$

$$[L_m, \phi] = \oint \frac{dz}{2\pi i} z^{m+1} \left[ \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{(z-w)} \right]$$

$(L_n|0\rangle = 0$ $n \geq -1$ for a unitary Theory $H = \mathrm{span}(L_n|0\rangle$ $n \leq -2)$

$$= (m+1) w^m h \phi(w) + w^{m+1} \partial_w \phi(w)$$

$\Rightarrow$ $[L_m, \phi(0)] = 0$ $\forall$ $m > 0$ if $m = 0$ $= h\phi(0)$

$\Rightarrow$ $L_0 |h\rangle = L_0 \phi(0)|0\rangle = \phi(0) L_0 |0\rangle + h\phi(0)|0\rangle$

$L_0|h\rangle = h|h\rangle$

$$L_n|h\rangle = L_n\phi(0)|0\rangle = -\phi(0)L_n|0\rangle$$

$$L_n|h\rangle = 0 \quad \forall\, n>0$$

A State satisfying

$$L_0|h\rangle = h|h\rangle \;;\; L_n|h\rangle = 0 \;\forall\, n>0$$

is a Highest Weight State.

To obtain The descendents we act with The Virasoro generators!

i.e $\quad L_{-n}|h\rangle = L_{-n}\phi(0)|0\rangle$

$$= \frac{1}{2\pi i}\oint dz\, z^{1-n}\, T(z)\phi(0)|0\rangle$$

$\therefore$ we can in general define The decesents of $\phi$ as

$$\phi^{-n}(w) \equiv (L_{-n}\phi)(w) = \frac{1}{2\pi i}\oint dz\, \frac{T(z)\phi(w)}{(z-w)^{n-1}}$$

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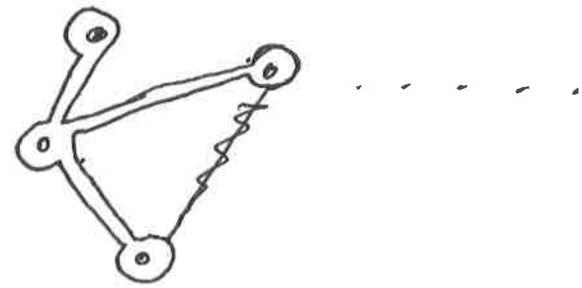
Finally Consider


We Perform a conformal transformation inside This region by integrating $\epsilon(z)T(z) = J(z)$ around The Contour.

$$\langle \oint dz\, J(z)\, \phi(w_1)\phi(w_2) \cdots \phi(w_n)\rangle$$

deform the contour

This becomes



$$= \sum_i \langle \phi(w_1) \cdots \oint dz\, J(z)\phi(w_i) \cdots \phi(w_n)\rangle$$

$$= \sum_i \langle \phi(w_1) \cdots \delta\phi(w_i) \cdots \phi(w_n)\rangle$$

$$= \sum_i \langle \phi(w_1) \cdot \cancel{\oint dz} \left(-\frac{h_i}{(z-w_i)^2} + \frac{\partial_{w_i}}{(z-w_i)}\right) \cdot \phi(w_n)\rangle$$

$$\Rightarrow \langle \oint \frac{dz}{2\pi i} \epsilon T\, \phi(w_1) \cdots \phi(w_n)\rangle - \sum_i \langle \phi(w_1) \cdots \oint \frac{dz}{2\pi i}\left(\frac{-h_i}{(z-w_i)^2} + \frac{\partial_{w_i}}{(z-w_i)}\right) \cdots \phi(w_n)\rangle = 0$$

$$\Rightarrow \oint \frac{dz}{2\pi i} \epsilon(z) \left[\langle T(z)\phi(w_1) \cdots \phi(w_n)\rangle - \sum_i \langle \phi(w_1) \cdots \left(\frac{-h_i}{(z-w_i)^2} + \frac{\partial_{w_i}}{(z-w_i)}\right) \cdots \phi(w_n)\rangle\right] = 0$$

$$\Rightarrow \quad \langle T(z)\phi(w_1) \cdots \phi(w_n)\rangle = \sum_i \langle \phi(w_1) \cdots \left(\frac{-h_i}{(z-w_i)^2} + \frac{\partial_{w_i}}{(z-w_i)}\right) \cdots \phi(w_n)\rangle$$

This is the conformal Ward ID!

This must hold so that the local conformal ~~system~~ symmetry survives quantization!

~~Thus~~ There is a set of corresponding statements for the Global transformations!