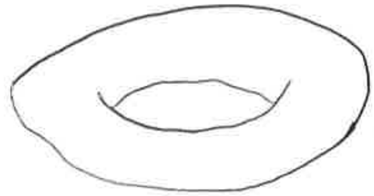
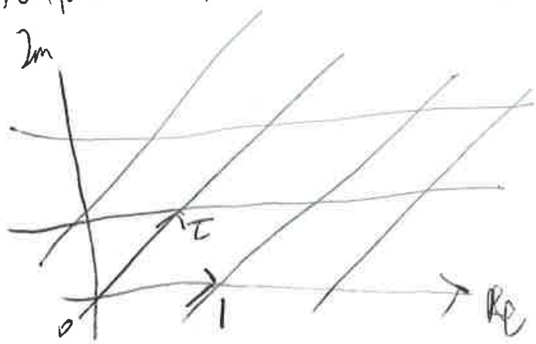
1. Intuitively one would expect all the states in a theory to contribute to loop diagrams. Hence loop diagrams should be a useful tool. So it's important to study conformal field theory on the simplest loop diagram, the torus.



The torus is a cylinder whose ends have been sewn together. The most convenient mathematical description of the torus is in terms of the complex plane modulo a lattice.



All points in the complex plane that differ by a linear combination of the two basic lattice vectors are considered identical. Identification along the real axis has the effect of rolling up the complex plane to a cylinder, then identification the vector labelled τ rolls up the cylinder to a torus.

This picture has a lot of symmetries in it:

1. Two-dimensional general coordinate invariance: straighten the coordinates to get a lattice.
2. Rotational invariance: make one direction point along the real axis.
3. Translation invariance: put a point of the lattice at the origin.
4. Global scale invariance: make the horizontal lattice spacing equal to 1. This means the entire lattice is described by one complex number, τ, which is chosen in the upper half plane.

2. The partition function:

Consider the path integral:

$$\int D\phi \, e^{-S_E(\phi)}$$

S_E is the Euclidean action of a given field configuration on the torus. The integral is over all field configurations. If we have a Lagrangian description of a conformal field theory, ϕ stands for all fields in the theory. However, in many cases such a Lagrangian formulation is either not available, or not practically usable. So it is more convenient to express the path-integral in terms of the Hamiltonian of the theory. The Hamiltonian is $L_0+\bar{L}_0$.

In ordinary quantum dynamics (0+1 dimensions) one have:

$$\underbrace{\int_{PBC}}_{\text{Periodical boundary condition}} Dq \, e^{-S_E(q)} = \mathrm{Tr}\, e^{-\beta H}$$

The integral is over all paths $q(t)$ that start at $t=0$ and end at $t=\beta$, with $q(0)=q(\beta)$ (PBC).

In the lattice description of the torus, if $\mathrm{Re}\,\tau=0$:

$$\int D\phi \, e^{-S_E(\phi)} = \mathrm{Tr}\, e^{-2\pi \mathrm{Im}\,\tau H}$$

The factor 2π appears because the torus has a periodicity 1 rather than 2π along the x' direction. As here we scale up the entire lattice by a factor 2π, so we get $2\pi\tau$ instead of τ.

~~If $\mathrm{Re}\,\tau \neq 0$, we have to twist the torus before gluing it together again.~~

If $\mathrm{Re}\,\tau\neq 0$, a shift by an amount $2\pi \mathrm{Re}\,\tau$ is achieved by the operator:

$$e^{iP(2\pi \mathrm{Re}\,\tau)}$$

with P the momentum operator

~~The operators H and p~~

The correct result is obtained by integrating this shift operator in the trace:

$$\int D\phi\, e^{-S_{\tau}(\phi)} = \mathrm{Tr}\, e^{-2\pi \mathrm{Im}\tau H} e^{iP(2\pi \mathrm{Re}\tau)}$$

H and p are the time and space translation operators on the cylinder.

3. The cylinder vs. the Riemann sphere.

We now examine how the energy-momentum tensor defined on the complex plane is related to that on the cylinder.

We need the transformation of the energy momentum tensor under conformal transformations.

Infinitesimal conformal transformations are generated by

$$Q_{\epsilon} = \frac{1}{2\pi i}\oint dz\, \epsilon(z) T(z)$$

The infinitesimal conformal transformation of the energy momentum tensor is:

$$\delta_{\epsilon} T(w) = [Q_{\epsilon}, T(w)] = \frac{1}{2\pi i}\oint dz\, \epsilon(z)\, T(z) T(w)$$

We insert the operator product of $T(z)$ and $T(w)$, Card part just.

$$\delta_{\epsilon} T(w) = \epsilon(w)\partial T(w) + 2\partial\epsilon(w) T(w) + \frac{c}{12}\partial^3\epsilon(w)$$

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w)$$

c = ~~the number of ϕ in the theory~~. central charge in Virasoro algebra.

The global form of this trasformation is

$$T(w) \to (\partial f)^2\, T(f(w)) + \frac{c}{12} S(f, w)$$

with $S(f,z) = \dfrac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}$ (schwartzian derivative)

Apply this to the map from the plane to the cylinder. The map we will use is $w = e^{iz}$, w is the plane coordinate and z the cylinder coordinate.

We have:

$$T_{cyl}(z) = [\partial_z w(z)]^2 T(w(z)) + \frac{c}{12} S(w,z)$$

$$= -[w^2 T(w) - \frac{c}{24}]$$

and $T(w) = \sum_n w^{-n-2} L_n$ (the mode expansion for $T(w)$)

So $T_{cyl}(z) = -[\sum_n e^{-inz}(L_n) - \frac{c}{24}]$

and analogously for the anti-holomorphic component.

We'll also need to find the precise definition of H and P.

$$H = \frac{1}{2\pi}\int dx' T^M_{00} = \frac{1}{2\pi}\int dx' T^E_{22} = -\frac{1}{2\pi}\int d\mathrm{Re}\,z\,[T^{cyl}(z) + \bar{T}^{cyl}(\bar{z})]$$

$$P = \frac{1}{2\pi}\int dx' T^M_{01} = \frac{1}{2\pi}\int dx' (-i T^E_{21}) = -\frac{1}{2\pi}\int d\mathrm{Re}\,z\,[T^{cyl}(z) - \bar{T}^{cyl}(\bar{z})]$$

$$\therefore\ H = L_0 - \frac{c}{24} + \bar{L}_0 - \frac{\bar{c}}{24}$$

$$P = (L_0 - \frac{c}{24}) - (\bar{L}_0 - \frac{\bar{c}}{24})$$

It shows the possibility that the holomorphic and anti-holomorphic components have different central charges.

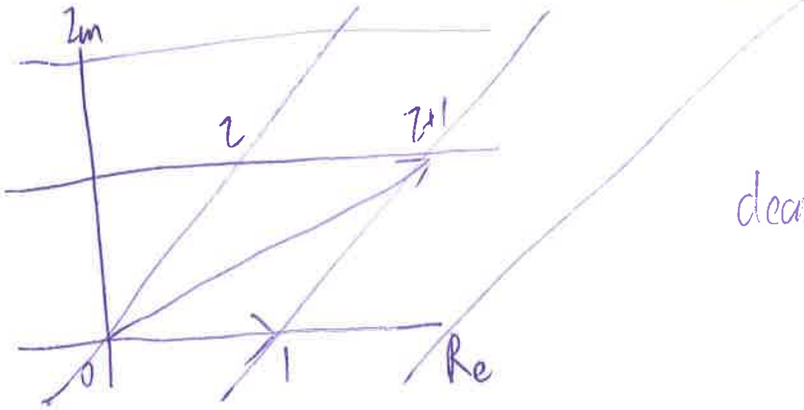
(H and P do not generate space and time translations on the plane, which are generated by L_{-1} and $\bar{L}_{-1}$.)

H is proportional to the dilatation operator on the plane and P is proportional to the rotation operator.

$$\therefore\ \int D\phi\, e^{-S_E(\phi)} = \mathrm{Tr}\, e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau}(\bar{L}_0 - \frac{\bar{c}}{24})} \equiv P(\tau, \bar{\tau})$$

4. Modular invariance

We have defined the torus in terms of a lattice. This lattice was defined by two basis vectors, "1" and "τ" in the complex plane. But we could select different basis vectors.



The choice 1 and τ+1 clearly describes the same lattice.

The set of such transformations of the torus forms a group called the modular group. Two elements of the group are:

$$T: \tau \to \tau+1$$

$$S: \tau \to -\frac{1}{\tau}$$

These two transformations generate the entire group. The most general modular transformation has the form

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad a,b,c,d \in \mathbb{Z}, \; ad-bc=1.$$

This group is isomorphic to $SL_2(\mathbb{Z})/\mathbb{Z}_2$. Also ~~are~~ the modular transformations satisfy

$$(ST)^3 = S^2 = 1.$$

5. Virasoro characters

The partition function can be expressed in terms of primary fields and descendants:

$$P(\tau,\bar{\tau}) = \sum_{i,j} M_{ij} \chi_i(\tau) \chi_j(\bar{\tau})$$

i and j label a certain highest weight states $|i,j\rangle$. M_{ij} is a nonnegative integer. χ are the Virasoro characters of the representation.

$$\chi_i(\tau) \equiv \mathrm{Tr}_{\text{descendants of } i}\, e^{2\pi i \tau (L_0 - \frac{c}{24})}$$

The trace is over all positive norm states in the highest weight representation labelled i.

6. Modular transformations of the characters.

The multiplicities M_{ij} must make the partition function modular invariant.

~~The characters transform as~~

To verify if a partition function written in terms of characters is modular invariant, we need to know how the characters transform.

For transformation T:

$$\chi_i(\tau+1) = e^{2\pi i (h_i - \frac{c}{24})} \chi_i(\tau)$$

or in matrix form:

$$\chi_i(\tau+1) = \sum_j T_{ij} \chi_j(\tau)$$

T is a diagonal matrix of phases.

For transformation S:

$$\chi_i(-\tfrac{1}{\tau}) = \sum_j S_{ij} \chi_j(\tau)$$

7. Conditions for modular invariance

$$[M, T] = [M, S] = 0$$

with $M_{ij} \in \mathbb{Z}$

It is usually supplemented with the additional physical requirement that the vacuum is present in the theory and is unique. If we label the vacuum by "0", then we have

$$M_{00} = 1$$

8. The diagonal invariant

These conditions have a trivial solution

$$M_{ij} = \delta_{ij}$$

It's called the diagonal invariant,

Some remarks: It is not hard to construct partition functions that are not modular invariant, but usually those are rejected. They can not correspond to well-defined two-dimensional theory on the torus.
An important consequence of modular invariance is that operator products are local; ~~This implies~~ that it has no branch cuts as a function of $z-w$.

9. Fusion rules.

For the three-point function

$$G^{(3)}_{ijk} = \langle 0|\phi_i(z_1)\,\phi_j(z_2)\,\phi_k(z_3)|0\rangle$$

It has $G^{(3)}_{ijk}(z_{12}, z_{23}, z_{31}) = C_{ijk}\, z_{12}^{h_3-h_1-h_2}\, z_{23}^{h_1-h_2-h_3}\, z_{31}^{h_2-h_3-h_1}$

The coefficients C_{ijk} satisfy certain selection rules.
~~The selection~~ i, j and k label fields $\phi_i(z,\bar{z})$. Label the fields as $\phi_{i,\bar{i}}$.

The selection rules imposed by the Virasoro algebra are called fusion rules and written as:

$$[i]\times[j] = \sum_k N_{ij}^{\ k}[k]$$

$[i]$ ... label representations of the Virasoro ~~ateg~~ algebra.
N_{ijk} is a set of non-negative integers.

If $N_{ij}^{\ k}$ vanishes, then $C_{i\bar{i},j\bar{j},k\bar{k}}$ vanish. If it does not vanish, the corresponding C is allowed and usually is non-zero.

10. Verlinde formula.

There is a relation between the fusion rule coefficients and the matrix S, discovered by E. Verlinde.

$$N_{ijk} = \sum_n \frac{S_{in} S_{jn} S_{kn}}{S_{0n}}$$

If we see N_{ijk} as a collection of matrices $(N_i)_{jk} = N_{ijk}$, then

$$(S^{+} N_i S)_{pq} = \left(\frac{S_{iq}}{S_{0q}}\right) \delta_{pq}$$

The ratios $\lambda_i^{(n)} \equiv \frac{S_{in}}{S_{0n}}$ are called the (generalized) quantum dimensions of the field i. So the Verlinde formula is:

$$\lambda_i^{(n)} \lambda_j^{(n)} = \sum_k N_{ijk} \lambda_k^{(n)}$$

11. Higher genus partition functions.

On a surface with n handles we can define a basis of homology cycles $a_i, b_i, i=1, \cdots n$:

On can choose a set of n holomorphic 1-forms on the surface and normalize them so that: $\int_{a_i} \omega_j = \delta_{ij}$

Then the integral along the b cycles defines the period matrix Ω_{ij}

$$\int_{b_i} \omega_j = \Omega_{ij}$$

This is the higher genus generalization of τ. ~~The~~ In the lattice picture of the torus, the a-cycle is the line from 0 to 1, and the b cycle the line from 0 to τ. The holomorphic 1-form is dz. ~~Ex~~ And

$$\int_a dz = \int_0^1 dz = 1 \text{, which is properly normalized. So}$$

$$\Omega_{11} = \int_b dz = \int_0^\tau dz = \tau$$

The modular transformation S corresponds to the mapping $a \to b$ and $b \to a$, while T is ~~to rp~~ replace b by b+a, without changing a.

The higher genus generalization of the modular transformation is:

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} \quad \text{with}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z})$$

~~Lecture on Con~~

Conformal Field Theory

A. N. Schellekens

A highest weight representation is a representation containing a state with a smallest value of L_0. For a highest weight state $|h\rangle$

$$L_n |h\rangle = 0 \quad \text{for } n \geq 1$$

$$L_0 |h\rangle = h |h\rangle$$

$$L_n = \frac{1}{2\pi i} \oint dz\, z^{n+1} T(z), \quad T(z) = \sum_n z^{-n-2} L_n$$

Virasoro algebra: $[L_n, L_m] = (n-m) L_{m+n} + \frac{c}{12} n(n^2-1) \delta_{n,-m}$