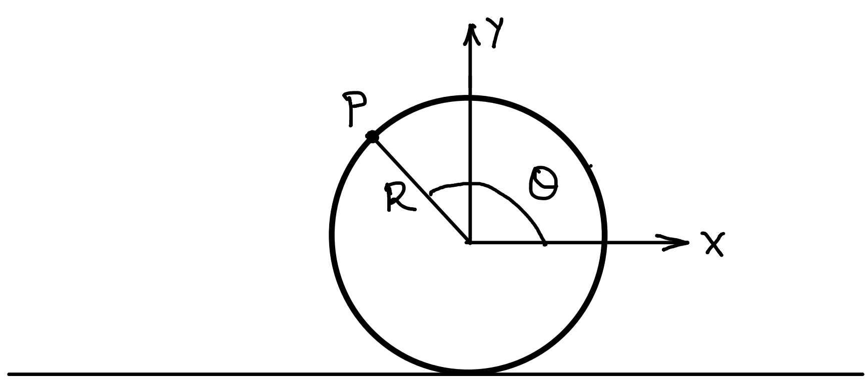
University of Alabama Department of Physics & Astronomy
Graduate Qualifying Examination
Classical Mechanics

8 January 2018

General Instructions

- No reference materials are allowed.
- Do *all* your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your *assigned number* and the subject.
- Turn in the questions for each part with the answer booklet.
- 150 minutes are alloted for this part.
- No electronic devices are allowed. In particular, handheld computers, PDAs, and cell phones are explicitly prohibited.
- Answer 5 out of the 6 problems and clearly indicate the problems you wish to be graded.

1. An object with mass m falling vertically under the influence of gravity through a viscous medium is subject to a resistive force that is proportional to the square of its speed, $F_R = cv^2$, where c is a constant. When dropped from rest, the speed of the object is described by the equation $v(t) = v_t tanh(t/\tau)$, where v_t is the terminal speed and τ is the characteristic time.
   a. Find an expression for the characteristic time in terms of only the mass, m, the acceleration due to gravity, g, and the proportionality constant, c.
   b. Verify that your equation is dimensionally correct.
2. A one-dimensional system of mass m is subject to a linear restoring force with force constant k and a frictional force proportional to velocity; denote the proportionality constant by b.
   a. If the system is initially at rest and displaced distance d from equilibrium, find the condition that must be satisfied so that the system does not oscillate when released.
   b. Suppose that the system is now subjected to a harmonic driving force: $F=F_0 \cos\omega t$. Find the amplitude of the velocity. Hint: The solution will have the form $x=A\cos(\omega t+\delta)$
   c. Find the driving frequency ω at which the velocity amplitude is maximum.
3. A hoop of radius R and uniform density is rolling without slipping on a horizontal surface while its center of mass is undergoing constant acceleration a towards the right (see the sketch below). Consider the instant at which the speed of the hoop as measured by a stationary observer is v.
   a. As measured by an observer in whose reference frame the center of mass of the hoop is at rest, find the acceleration of a point on the hoop as a function of θ, v, a, and R and the unit vectors $\hat{x}$ and $\hat{y}$.
   b. As measured by a stationary observer, find the point on the hoop that has the greatest acceleration and give the magnitude and direction of the acceleration relative to the horizontal.



4. The *effective* potential governing the radial motion of a satellite of mass m moving in the neighborhood of a black hole is

$$V_{eff} = \frac{1}{2}mc^2\left[-\frac{R}{r} + \frac{\ell^2}{r^2} - \frac{R\ell^2}{r^3}\right]$$

In the above expression R is the Schwarzschild radius and $\ell = \dfrac{L}{mc}$ where L is the satellite's angular momentum.

   a. Show that for a given $\ell > \sqrt{3}R$ there are possible circular orbits and find their radii in terms of ℓ and R.
   b. Of the orbits found in the previous part, show that only one is stable.
   c. Show that the satellite cannot have a stable circular orbit with radius less than 3R.

5. A speed governor on an old-fashioned steam engine consists of two massive balls that are connected to a rotating rod by light rods. Derive an expression for the angle that the rods make with the vertical as a function of angular velocity ω. Denote the distance from the center of each mass to the opposite end of the connecting rod by *L*, and denote the mass of each ball by *m*.

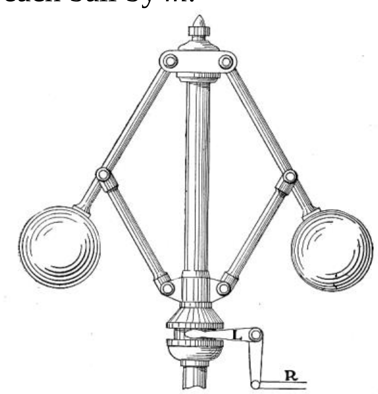
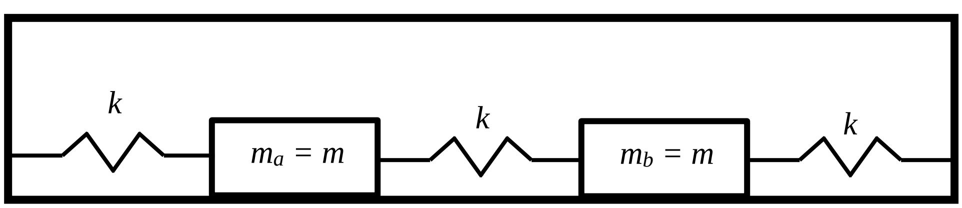


FIG. 15—A FLY-BALL GOVERNOR

6. Consider two identical masses m resting on a frictionless surface and connected by three identical light springs with spring constant k as shown below. Define x_a and x_b as the displacements of the masses from their respective equilibrium positions.
   a) Find the Lagrangian of the system in terms of the variables $u = x_a + x_b$ and $v = x_a - x_b$.
   b) Find the corresponding equation of motion and the resulting frequencies of oscillations.



University of Alabama Department of Physics & Astronomy
Graduate Qualifying Examination
Electricity and Magnetism

9 January 2018

General Instructions

- No reference materials are allowed.
- Do *all* your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your *assigned number* and the subject.
- Turn in the questions for each part with the answer booklet.
- 150 minutes are alloted for this part.
- No electronic devices are allowed. In particular, handheld computers, PDAs, and cell phones are explicitly prohibited.
- Answer 5 out of the 6 problems and clearly indicate the problems you wish to be graded.

1: Point dipole

A point dipole, $\mathbf{p} = p_0\hat{\mathbf{z}}$, is located at the center of a linear, isotropic, homogeneous sphere with radius R and permittivity $\varepsilon = \kappa\varepsilon_0$. Find the electric potential outside the sphere.

First few Legendre Polynomials

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

Associated Legendre Polynomials

$$P_{\mathrm{n}}^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}\left(P_{\mathrm{n}}(x)\right)$$

2: Shell with fixed charge density

The charge density of a hollow spherical shell of radius R has the form $\sigma(\theta) = \sigma_0\sin^2\theta$.

(a) Find the potential inside the sphere.

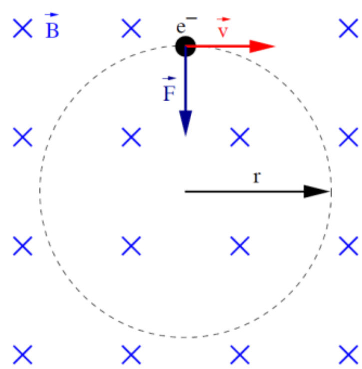
(b) Find the potential outside the sphere.

3: Relativistic Cyclotron

A relativistic electron (rest mass m_0 and charge $-e$) moving in the xy-plane with speed v is subjected to a magnetic field with magnitude B in the z direction. Since the magnetic field does no work, the electron travels in a circular orbit with angular frequency ω_c and radius r_c.

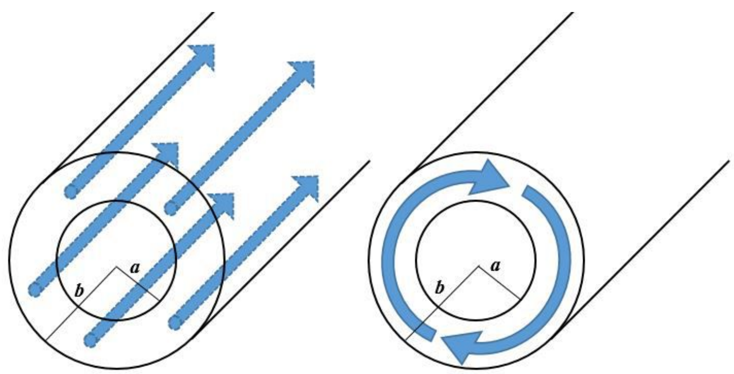
(a) Find the angular frequency in terms of $\gamma = 1/\sqrt{1-v^2/c^2}$, m_0, e and B.

(b) Find the radius of the orbit in terms of γ, v, m_0, e and B.



4: Cylinders of Current

For each of the geometries shown in figures (i) and (ii) below find the magnetic field ($\vec{B}$) everywhere in space and then sketch the plot of |B| as a function of the distance from the central axis. Assume that the current density $\vec{J}$ is uniform within the shell. Express your answer in terms of the total current I.



(i) $\vec{J} = j_o\hat{z}$ for $a < \rho < b$

(ii) $\vec{J} = j_o\hat{\phi}$ for $a < \rho < b$

5: Concentric Spherical Shells

Three concentric spherical metallic shells have radii c>b>a. They are initially charged with q_c, q_b and q_a, respectively. Finally, the inner shell is grounded. Find the resulting change in potential of the outermost shell.

6: Reflection and Transmission

A plane-polarized EM wave with electric field $\vec{E_I} = \hat{x}E_I e^{i(k_1 z - \omega t)}$ is traveling in a transparent, nonmagnetic medium with index of refraction n_1. It is normally incident on an interface at $z = 0$ with a transparent, nonmagnetic medium with index of refraction n_2. At the interface, the reflected and transmitted electric fields are $\vec{E_R} = -\hat{x}E_R e^{i(-k_1 z - \omega t)}$ and $\vec{E_T} = \hat{x}E_T e^{i(k_2 z - \omega t)}$, respectively with $k_1 = n_1\frac{\omega}{c}$ and $k_2 = n_2\frac{\omega}{c}$.

(a) Find the transmission coefficient, $t_{12} = \frac{E_T}{E_I}$, in terms of the indices of refraction.

(b) Find the reflection coefficient, $r_{12} = \frac{E_R}{E_I}$, in terms of the indices of refraction.

University of Alabama Department of Physics & Astronomy Graduate Qualifying Examination
Quantum Mechanics

21 August 2018

General Instructions

1. **Sign the role sheet; the number beside your name will be used as your identification.**
2. **Answer booklets are numbered, and the graders will not be given the names.**
3. **DO NOT put your name on any of the materials.**
4. **Seats are assigned, so take the seat number indicated on your answer booklet.**
5. **Answer 5 out of 6 of the problems.**
6. **Clearly write on the inside cover of the answer booklet the problems you want to be graded.**
7. **150 minutes are allotted for this exam.**

- No reference materials are allowed.
- No scratch paper is allowed.
- Do all your work in the corresponding answer booklet.
- On the cover of each answer booklet write only your assigned identification number.
- Turn in the question sheets and all answer booklets at the end of the exam.
- No electronic devices are allowed in your possession, including calculators, computers, and cell phones.
- If you inadvertently brought your cell phone, please give it to the proctor to keep until you finish the exam.

1. Say that a hydrogen atom has orbital angular momentum quantum number $\ell = 1$. To obtain the total angular momentum $\vec{J}^{TOT}$ of the atom one should also include the spins $\vec{S}^{(1)}$ and $\vec{S}^{(2)}$ of the electron and the proton.

   (a) (10 points) What then are all the possible eigenvalues for J_z^{TOT}, and with what multiplicities do they occur?

   (b) (10 points) What are all the possible eigenvalues for $(J^{TOT})^2$, and with what multiplicities do they occur?

2. Recall that the energy eigenfunctions and eigenvalues for one particle in the harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$ are

$$\phi_n(x) = \Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\Big(\sqrt{\frac{m\omega}{\hbar}}\, x\Big) e^{-\frac{m\omega}{2\hbar} x^2} , \qquad E_n = \Big(n + \frac{1}{2}\Big)\hbar\omega$$

where H_n are the Hermite polynomials, the first two given $H_0(\xi) = 1$ and $H_1(\xi) = 2\xi$.

(a) (5 points) Give an explicit expression for the ground state wavefunction $\psi(x_1, x_2)$ of two identical bosons in the harmonic oscillator potential at a time $t = 0$ and at a later time $t = t'$.

(b) (5 points) Repeat for two identical fermions, assuming they are in identical spin states.

(c) (5 points) What is the ground state energy of *three* identical bosons trapped in a harmonic oscillator potential?

(d) (5 points) Repeat for three identical spin–$\frac{1}{2}$ particles trapped in a harmonic oscillator potential.

3. Suppose that in an experiment, there are two orthonormal quantum states A and B, which are represented respectively by

$$|\psi_A\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad |\psi_B\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

In this system, the normalized energy eigenstates are given by

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the corresponding eigenvalues are E_1 and $E_2 = E_1 + E_0$ with $E_0 > 0$, respectively.

(a) (7 points) If a wave function at time $t = 0$ is detected as $|\psi(0)\rangle = |\psi_A\rangle$, express the wave function $|\psi(t)\rangle$ at a time $t > 0$ in terms of $|\psi_1\rangle$ and $|\psi_2\rangle$.

(b) (7 points) Find a probability to detect the state as $|\psi_B\rangle$ at a time $t > 0$, when a wave function at time $t = 0$ is detected as $|\psi_A\rangle$.

(c) (6 points) What is the maximum probability to detect the state as $|\psi_B\rangle$, if a wave function at time $t = 0$ is detected as $|\psi_A\rangle$?

4. The electron of a hydrogen atom is in an energy eigenstate described by the wave function

$$\psi(\vec{r}) = A\left(\frac{r}{a_0}\right)^2 e^{-\frac{r}{3a_0}} \cos\theta \sin\theta \cos\phi$$

where a_0 is the Bohr radius and A is a constant.

(a) (7 points) If the radial position of the electron were measured, what would be the most probable result?

(b) (7 points) If the magnitude of the orbital angular momentum were measured, what would be the outcome? Explain.

(c) (6 points) What would be the possible outcome(s) of measuring the z–component of the orbital angular momentum? Explain.

Given:

$$\hat{H} = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)\right] - \frac{e^2}{4\pi\epsilon_0 r}$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)\right] \qquad \hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\int_0^\infty dx x^n e^{-x} = n!$$

5. We consider the so-called asymmetric harmonic oscillator in two dimensions defined by the following Hamiltonian,

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right),$$

where ω_x and ω_y are two different frequencies ($0 < \omega_x < \omega_y$).

(a) (7 points) By expressing the energy eigenfunction of the system as $\psi(x, y) = u(x)v(y)$, decompose the time-independent Schrödinger equation into two independent parts.

(b) (7 points) Give the energy of the ground state.

(c) (6 points) Give the energy of the first excited state.

6. A particle of mass m moves in one dimension under the influence of the potential:

$$V(x) = \begin{cases} \infty & x < -a \\ 0 & -a < x < -b \\ V_0 & -b < x < b \\ 0 & b < x < a \\ \infty & x > a \end{cases}$$

(a) (12 points) Suppose $|V_0| \ll \frac{\hbar^2}{mab}$. Use first–order perturbation theory to find an expression for the ground state energy..

(b) (8 points) Suppose instead that V_0 is large and positive such that $V_0 \gg E_0$, where E_0 denotes the ground state energy for this case. Sketch the wave functions for the lowest two energy states. Estimate E_0.

University of Alabama Department of Physics & Astronomy Graduate Qualifying Examination
Thermal Physics

22 August 2018

General Instructions

Answer 2 out of 3 of the problems. Clearly indicate on the inside cover of the answer booklet the problems you wish to be graded.

- No reference materials are allowed.
- Do all your work in the corresponding answer booklet.
- No scratch paper is allowed.
- On the cover of each answer booklet put only your assigned number and the part number/subject.
- Turn in the questions for each part with the answer booklet.
- 90 minutes are alloted for this part.
- No electronic devices are allowed. In particular, handheld computers, PDAs, and cell phones are explicitly prohibited.

1. Ideal Bose Gas.

Consider a collection of N identical, non-interacting, spinless Bose particles. Each particle has only two possible energy eigenstates, ψ_0 with energy $\varepsilon = 0$ and ψ_1 with energy $\varepsilon = \Delta$. The number of particles n_1 in the eigenstate ψ_1 can be used as the index for the many particle states of the system with $|n_0, n_1\rangle = |N - n_1, n_1\rangle$ and $E_{n_1} = n_1\varepsilon$ with possible values of $n_1 = 0,1,2, \ldots, N$ giving $N+1$ many particle states.

a) Find a close-form expression for the partition function $Z(N, T)$ using the Canonical Ensemble. (20 pts)

b) Find the probability $P(n)$ that n particles will be found in the eigenstate ψ_1. (15 pts)

c) Find the partition function $Z_d(N, T)$ that would apply if the N particles were distinguishable but still had the same two single-particle eigenstates as above. (15 pts)

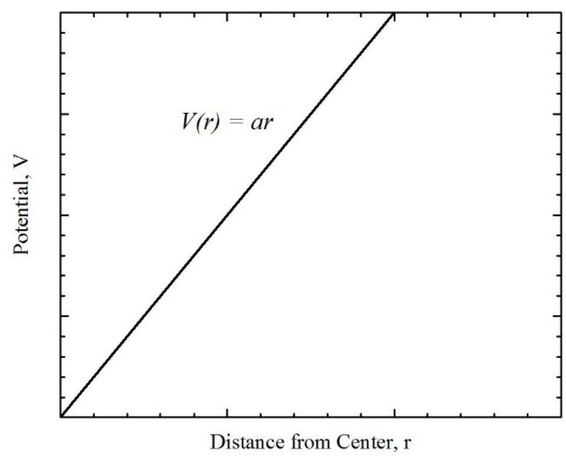
2. Neutral Atom Trap.

A gas of N indistinguishable classical non-interacting atoms is held in a neutral atom trap by a potential of the form $V(r) = ar = a\sqrt{x^2 + y^2 + z^2}$. The gas is in thermal equlibrium at temperature T.

a) Find the single particle partition function $Z_1 = \frac{1}{h^3}\int e^{-\beta H(r,p)} d^3r d^3p$ for a trapped atom with $H(r,p) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + ar$. Express your answer in the form $Z_1 = AT^{\alpha}a^{-\eta}$, that is find the prefactor A and the exponents α and η. HINT: $d^3r = r^2 sin\theta dr d\theta d\phi$ in spherical coordinates. (20 pts)

b) Find the entropy of the gas in terms of N, k, and Z_1. (15 pts)

c) The gas can be cooled if the potential is lowered reversibly by slowly decreasing a while no heat is allowed to be exchanged with the surroundings, $dQ = 0$. Under these conditions, find T as a function of a and the initial values T_0 and a_0. (15 pts)



Simplified potential function for the neutral atom trap.

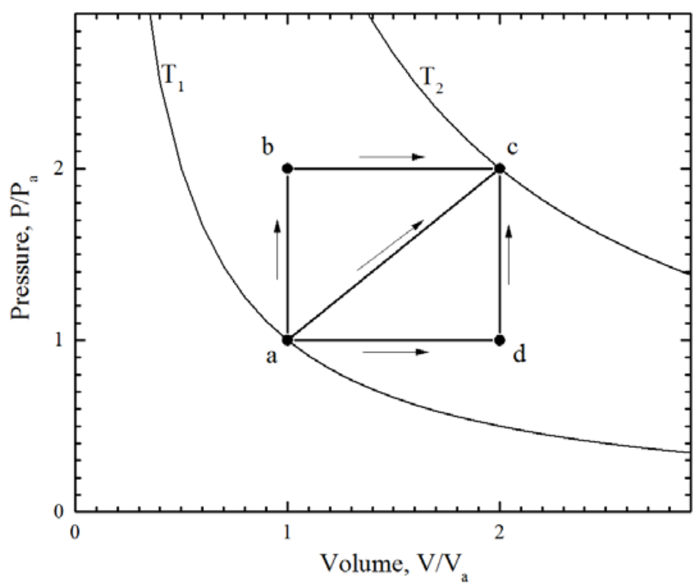
3. Heat Supplied to a Gas.

An ideal diatomic gas ($PV = NkT$) with heat capacity at constant volume, $C_v = \frac{5}{2}Nk$, is taken from point a to point c in the pressure-volume diagram along three possible paths as shown in the figure below. For each of the following questions assume that $\left(\frac{\partial U}{\partial V}\right)_T = 0$.

a) Find the heat capacity at constant pressure C_p in terms of N and k. (20 pts)

b) Compute the heat supplied to the gas along each of the three paths: *abc*, *adc*, and *ac* in terms of N, k, and T_1. (15 pts)

c) Find the "heat capacity" along path *ac*: $C_{ac} = \left.\frac{dQ}{dT}\right|_{ac}$. (15 pts)



Pressure-volume diagram showing three possible paths from a to c.