Talk 1: 4-3-2 Dimensional Ladder (And Talk 2 maybe)

Motivation: Relate a relatively easy to calculate "abelian" anomalies to "non-abelian" anomalies

Result: We define a particular gauge transformation

$$\mathcal{G} \equiv \{ g(\theta,\vec{x}) \mid g(0,\vec{x}) = g(2\pi,\vec{x}) = g(\theta,\vec{x}_0) = \mathbb{1}$$

$$g \in SU(2) \mid (\theta,\vec{x}) \in S^1 \times S^2$$

$$\underbrace{\nu_+ - \nu_- = \int_{S^2 \times S^2} \partial_\mu \overset{\text{"}F\wedge F\text{"}}{J^\mu}}_{\text{"abelian Part"}} = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \partial\theta \, \frac{\partial \omega}{\partial \theta} = N}_{\text{"non-abelian Part"}}$$

(3) Defining the non-abelian part of my talk

$$S = -\int d^2x\, \bar{\Psi}\, i\, \gamma^{\mu} (\partial_{\mu} + A_{\mu}) P_{+} \Psi$$

$\mu \in \{0,1\}$

External A_{μ} gauge field $\in$ [illegible] (3)

$= \sum A_{\mu}^{a} T^{a}$

over S^2

$P_{\pm} = \frac{1}{2}(1 \pm \gamma^3)$

Weyl of $r=2$ (doublet representation), $\gamma^{\nu} \in$ euclidean version of gamma matrices.

At a quick glance we can notice, classically, the current, $j^{\mu}_{a} = \bar{\Psi}\, i\, \gamma^{\mu} T_a P_{+}$. (Non-abelian current)

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On the level of operators however this current is not conserved. Working in "Euclidean" space we define a generating functional, $Z = e^{-\Gamma} = \int d\bar{\Psi}\, d\Psi\, e^{+S}$ (Γ = effective action). Seemingly, we have an operator $i\nabla_{+} = i\gamma^{\mu}(\partial_{\mu} + A_{\mu}) P_{+}$ $\therefore$ we hope express $e^{-\Gamma}$ as a product of eigenvalues of $i\nabla_{+}$ as $e^{-\Gamma} = \det(i\nabla_{+})$

3/2 contd • However, $\det(i\nabla_+)$ is ill defined because the $i\nabla_+$ operator maps positive chirality modes.

• We define a new operator $i\not{D}$ (acting on Dirac fermions)

$$i\not{D} = i\gamma^\mu(\partial_\mu + A_\mu P_+) = \begin{pmatrix} 0 & \not{\partial}_- \\ i\nabla_+ & 0 \end{pmatrix}$$

where before $i\nabla_+ = \begin{pmatrix} 0 & 0 \\ i\nabla_+ & 0 \end{pmatrix}$

• On a compact manifold we may construct a well-defined eigenvalue problem.

$$i\not{D}\psi_i = \lambda_i\psi_i \quad (i\not{D})^\dagger\chi_i = \bar{\lambda}_i\chi_i$$

• We may choose an orthonormal basis such that

$$\int_M \chi_i^\dagger\psi_j = \delta_{ij}$$

3/3 contd ($W \sim \cancel{Z}$)

• Eigenvalues of iD are not gauge invariant however because $(g^{\dagger} iD(A^g)g)\psi_i \neq (iD\psi_i)$ (II)

$= \lambda \psi_i$

• Nevertheless the spectrum of the absolute value of the eigenvalues is still gauge invariant. (Because of set to satisfy (II)

$D g^{-1}\psi = \lambda g^{-1}\psi$ . . .)

• We note $\det(iD)\det((iD)^{\dagger}) = \det(iD(iD)^{\dagger})$

$$= \det\left(\begin{pmatrix} 0 & i\partial_- \\ i\nabla_+ & 0 \end{pmatrix}\begin{pmatrix} 0 & i\nabla_- \\ i\partial_+ & 0 \end{pmatrix}\right) = \det\begin{pmatrix} i\partial_- i\partial_+ & 0 \\ 0 & i\nabla_+ i\nabla_- \end{pmatrix}$$

$$= \det(i\partial_- i\partial_+)\det(i\nabla_+ i\nabla_-)$$

$$= \det(i\partial_- i\partial_+)|\det(i\nabla)|$$

$\therefore$ Re $W[A]$ is gauge invariant

$\boxed{W \sim \cancel{Z}}$ $(\cancel{Z}[A^g] = \det(iD(A^g)) = \sqrt{\det i\nabla A}\; e^{iw(A,g)}$

anomaly

3) cont'd 4 (Mathematical 2nd time)

• Now consider $A^g = g^{-1}(A + d)g$

$dx^\mu A_\mu \quad dx^\mu \frac{\partial}{\partial x^\mu}$

$A^a_\mu T^a \quad \mathbb{1}$

$g = g(\theta, x)$

$g(\theta, x_0) = \mathbb{1}$

$g(0, x) = g(2\pi, x) = \mathbb{1}$

for spacetime $= S^m = S^{2l}$

we define g to map thus inducing another mapping

$S^1 \times S^{2l} \longrightarrow G \qquad A \longrightarrow A^g$

$(" S^{2l+1} =)$

∴ $\pi_{2l+1}(G)$ classifies g (and A^g) where

$\pi_{2l+1}(SU(N)) = \mathbb{Z}$ where $N \geq l+1$

Now further consider a family of smoothly connected gauge potential (parameterized by $t \in [0,1]$)

$A^{t, g(\theta)} = t A^{g(\theta)}$

3) contd

- $\forall \theta$ & $t = 1$ our anomaly is represented by an anomalous phase multiplying itself to a <u>gauge invariant</u> determinant

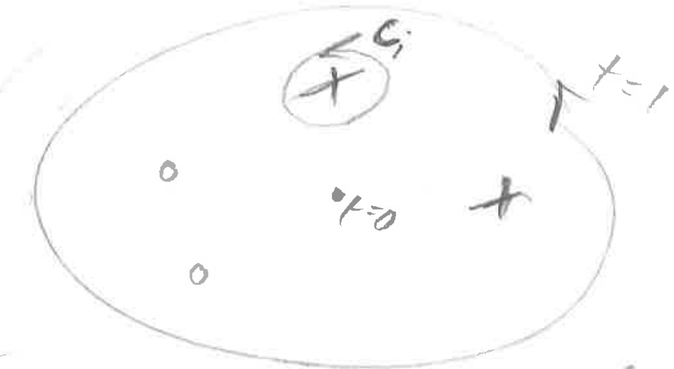
- The follow can be shown

$$W[A^{g(\theta+0)}] - W[A^{g(0)}] = -2\pi i N$$

where $N = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial w(A,\theta)}{\partial \theta}$

(a so called "Berry phase")

3) cont'd • Now let's let t be from 0 to 1.
The interpolation of A is not a gauge transformation $\therefore$ "$\mathrm{Det}(D[A^{t,\gamma}]) = 0$"
for some value of t.



From complex analysis we may relate N to the sum of the residues of ω inside our "Disk" $\cong m$.

$$M_{tot} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \, \frac{\partial}{\partial\theta}\left(\omega(A,\theta)\right)$$

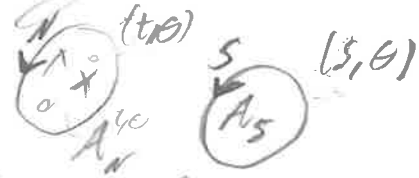
$$= \frac{1}{2\pi i}\oint_{S^1} \frac{\partial}{\partial\theta}(i\omega) = \sum_i \frac{1}{2\pi}\oint_{C_i} \frac{\partial\omega}{\partial\theta}$$

$$= \sum_i M_i \quad \left(= \mathrm{ind}\, D_{M+2} = N \right)$$

3) ^{7} cont'd: At this point we define a "family" of gauge fields on a 2-D disk ($=D^2$) w/ $A^{t,\theta}$

Our goal now is to prove the following identifications the $\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial w}{\partial \theta} = N = \sum_i m_i = \text{ind}\,(\nabla_{m+2})$

• Defining another disk, we may "sew" these disks such that they make the hemispheres of a $S^2 \times S^m$ sphere.



• On the "new" manifold we define new A's

$$A_N = A^{t,\theta} + g^{-1} d_\theta g$$

• Formally defining A globally we have

$$\mathbb{A}_{N,S} = (0,\, 0,\, (A_{N/S})_\mu)$$

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• 3) ~~Identify~~ • We see that there exist $\mathbb{A}$ on our constructed $S^2 \times S^m$ manifold globally. $S^2 \times S^m$ is a compact, boundaryless manifold equipped with a $\mathbb{A}$ gauge field and a $\nabla_{m+2}[\mathbb{A}]$ Dirac Operator. ∴ AS theorem yields

• $\nu_+ - \nu_- = \int_{S^2 \times S^m} ch_{l+1}(\mathbb{F})$

Nakahara $= (-1)^l \left(\frac{i}{2\pi}\right)^{l+1} \frac{l!}{(m+1)!} \int_{S^1 \times S^m} tr(g^{-1} \Delta g)^{m+1}$

$\left(= \left(\frac{1}{2\pi}\right) \int_0^{2\pi} d\theta \frac{\partial \omega}{\partial \theta} = N \right)$, $Q_2^1(\omega, A, \mathbb{F})$

$= \left(\frac{i}{2\pi}\right)^2 tr(\omega dA)$

shown in A.Ginsparg 1985 paper, see next page.

$\Delta = d + d_\theta + d_t$

$ch_{l+1}(\mathbb{F}) = \frac{1}{(l+1)!} tr\left(\frac{i\mathbb{F}}{2\pi}\right)^{l+1}$