Fujikawa PI Method to Chiral Anomaly

- Fall 18
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References

- Anomalies in QFT – Bertlmann, pg 249, 253
- Fujikawa. Phys. Rev. D21, 2848 (1980) ; Phys. Rev. lett. 42 (1979) 1195–1198
- QFT – Srednicki, pg 472

- Recall the basic features of the chiral anomaly

** Some definitions: This talk is mainly carried out using the euclidean metric (Wick transform)

- $x^0 = -ix^4$, $\partial_0 = i\partial/\partial x^4 = i\partial_4$, $\gamma^0 = -i\gamma^4$, $A_0 = iA_4$ and $g^{\mu\nu} = -\delta^{\mu\nu}$ $= \mathrm{diag}(-1,-1,-1,-1)$
- $\not{D} = \gamma^\alpha D_\alpha = \gamma^\alpha(\partial_\alpha + A_\alpha) \rightarrow \gamma^4 D_4 + \gamma^i D_i = \gamma^\alpha g_{\alpha\beta} D^\beta = -\gamma^\beta D^\beta$

Hermitian $\not{D}^\dagger = \not{D}$

- $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^4\gamma^1\gamma^2\gamma^3$

and $\{\gamma^\mu, \gamma_5\} = 0$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, $iA_\mu = g A_\mu^a(x) T^a$

- And under the chiral transformation (local)

$$\psi \rightarrow \psi' = e^{i\beta(x)\gamma_5}\psi$$
$$\bar\psi \rightarrow \bar\psi' = \bar\psi e^{i\beta(x)\gamma_5}$$

$$\Rightarrow \mathcal{L} \rightarrow \mathcal{L}' = \bar\psi'(i\not{D} - m)\psi'$$
$$= \mathcal{L}_0 - (\partial^\mu\beta(x))\bar\psi\gamma_\mu\gamma_5\psi - 2im\beta\bar\psi\gamma_5\psi$$
$$= \mathcal{L}_0 - (\partial^\mu\beta(x))J_\mu^5 - 2im\beta(x)P$$

- Implications for the classical action

$$(\mathcal{L}_0 = \bar\psi(i\not{D} - m)\psi\} + \tfrac{1}{2g^2}\mathrm{Tr}\{F^{\mu\nu}F_{\mu\nu}\})$$

$$S' = S + \int d^4x\, \beta(x)[\partial^\mu J_\mu^5 - 2imP(x)] \quad , \quad S = \int d^4x\, \mathcal{L}_0$$

- if the action is to remain invariant under the chiral transformation

$$S' \equiv S \Rightarrow \partial^\mu J_\mu^5 = 2imP$$ So we're done right?

Not quite, for the full quantum theory we need to consider the generating functional/PI.
That is what happens to $Z = \int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar\psi \exp[S]$ (euclidean)

- For now let's neglect the gauge (fixed background)

$$Z[A] = \int D\psi D\bar\psi \exp\left[\int dx\, \bar\psi(i\not{D} - m)\psi\right] = \det(i\not{D} - m)$$

- how will this transform under the chiral transform?

- Let's take a step back to analyze how Grassmann functions transform

Ex) Consider $\{\theta_i, \theta_j\} = 0$ $(\theta_i^2 \equiv 0)$

and $\int d\theta_2 d\theta_1\, \theta_1\theta_2 = 1$, what happens if $\theta_i \rightarrow \theta_i' = X_{ij}\alpha_j$?

$$\theta_1'\theta_2' = (X_{11}\alpha_1 + X_{12}\alpha_2)(X_{21}\alpha_1 + X_{22}\alpha_2) = X_{11}X_{22}\alpha_1\alpha_2 + X_{12}X_{21}\alpha_2\alpha_1 + \phi \quad (\alpha_i^2 = 0)$$
$$= (X_{11}X_{22} - X_{12}X_{21})\alpha_1\alpha_2 = \alpha_1\alpha_2 \det(X)$$

So $\int d\theta_2 d\theta_1' \theta_1' \theta_2' = \int d\alpha_2 d\alpha_1 J(X) \alpha_1 \alpha_2 \det(X) \overset{?}{=} 1$ (2)

$\Rightarrow$ for Grassmann integral the Jacobian transforms as $\boxed{\det(X)^{-1} = J}$

~~[illegible]~~

- Keeping this behavior of Grassmann variables in mind, let's precede by decomposing the Dirac fermions into eigenfunctions of the Dirac operator

$$\left.\begin{aligned}\Psi(x) &= \sum_n a_n \phi_n(x) = \sum_n a_n \langle x|n\rangle \\ \bar{\Psi}(x) &= \sum_m \phi_m^+(x)\bar{b}_m = \sum_m \langle m|x\rangle \bar{b}_m\end{aligned}\right\} \quad a_n + \bar{b}_m \text{ are Grassmann odd}$$

and
- $\not{D}\phi_n(x) = \lambda_n \phi_n(x)$ (Hermitian operator) $\lambda_n \in \mathbb{R}$
- $\int dx\, \phi_m^+(x)\phi_n(x) = \langle m|n\rangle = \delta_{mn}$
- $\sum_n \phi_n(y)\phi_n^+(x) = \langle y|x\rangle = \delta(y-x)$

- The Z transforms as

$$Z = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp\left[\int dx\, \bar{\Psi}(i\not{D}-m)\Psi\right] \to \int \prod_n da_n d\bar{b}_n \exp\left[\sum_{n,m}\int dx\, \phi_m^+(x)(i\lambda_n - m)\phi_n(x)\bar{b}_m a_n\right]$$

$$= \int \prod_n da_n d\bar{b}_n \exp\left[\sum_{n,m} \delta_{nm}(i\lambda_n - m)\bar{b}_m a_n\right] = \det(i\lambda_n - m) = \prod_n (i\lambda_n - m)$$

(functional det)

- Next, let's analyze the effect of the chiral transform on the decomposed eigenfunctions

$$\Psi'(x) = \sum_n a_n' \phi_n(x) = \sum_n e^{i\beta(x)\gamma_5} a_n \phi_n(x) \Rightarrow a_m' = \sum_n C_{mn} a_n$$

aside

$\longrightarrow$ To find a_n' use completeness relation

$$\sum_n \int a_n' \phi_m^+(x)\phi_n(x)dx = \sum_n \int \phi_m^+(x) e^{i\beta(x)\gamma_5} a_n \phi_n(x)dx$$

$$= \sum_n a_n'(\delta_{mn}) \Rightarrow a_m' = \left(\sum_n \int \phi_m^+(x) e^{i\beta(x)\gamma_5}\phi_n(x)dx\right) a_n = \sum_n C_{mn} a_n$$

$$\sum_n \int \phi_m^+(x)\phi_n^+(x)\bar{b}_m' = \sum_n \int \phi_n^+(x)\bar{b}_m e^{i\beta\gamma_5}\phi_n\, dx$$

$$\bar{b}_m' = \sum_n \bar{b}_n \int \phi_n^+(x) e^{i\beta(x)\gamma_5}\phi_m(x)dx = \sum_n \bar{b}_n C_{nm}$$

Similarly, for $\bar{b}_m' = \sum_n \bar{b}_n C_{nm}$

- $C_{mn} = \delta_{mn} + i\int dx\, \beta(x)\phi_m^+(x)\gamma_5\phi_n(x) + \mathcal{O}(\beta(x)^2)$
- Now we know for Grassmann, they transform as

$$\prod_n da_n' = (\det C)^{-1}\prod_n da_n, \quad \prod_m d\bar{b}_m' = (\det C)^{-1}\prod_m d\bar{b}_m$$

- and so

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = (\det C)^{-2} \mathcal{D}\psi\mathcal{D}\bar{\psi} = J(\beta)\,\mathcal{D}\psi\mathcal{D}\bar{\psi}$$

- Using the formula for det: $\det C = \exp\left[\mathrm{Tr}\{\ln C\}\right]$

$$J(\beta) = (\det C)^{-2} = \exp\left[-2\,\mathrm{Tr}\{\ln C\}\right] = \exp\left[-2\,\mathrm{Tr}\,\ln\{\delta_{mn} + i\int dx\,\beta(x)\,\phi_m^\dagger(x)\gamma_5\phi_n(x)\}\right]$$

where $\ln(1+\varepsilon) \approx \varepsilon + \mathcal{O}(\varepsilon^2)$

$$J(\beta) = \exp\left[-2i\int dx\,\beta(x)\,\underline{\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x)}\right]$$

but $\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) = \sum_n \gamma_5\,\phi_n^\dagger(x)\phi_n(x) = \mathrm{tr}\{\gamma_5\,\delta(0)\}$

isn't well defined! $\quad \infty$!

- To get around this seeming misshap, let's introduce a regulator

$$\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) = \lim_{M\to\infty} \sum_n \phi_n^\dagger(x)\gamma_5 \exp\left[\frac{-\lambda_n^2}{M^2}\right]\phi_n(x)$$

$$= \lim_{M\to\infty} \underbrace{\sum_n \phi_n^\dagger(x)\gamma_5\exp\left[-\frac{\not{D}^2}{M^2}\right]\phi_n(x)}_{\text{larger eigenvalues become more suppressed!}} \qquad \left(\underset{\text{Dirac op}}{\not{D}\phi_n = \lambda\phi_n}\right)$$

- Note that this is also still gauge invariant!

- Next, through the employ of some further "tricks" (Fourier transform)

$$\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) = \lim_{M\to\infty} \mathrm{Tr}\left\{\gamma_5\int\frac{d^4k\,d^4l}{(2\pi)^4}\sum_n \tilde{\phi}_n^\dagger(l)e^{-ilx}\gamma_5\exp\left[-\frac{\not{D}^2}{M^2}\right]e^{ikx}\tilde{\phi}_n(k)\right\}$$

$\quad\llcorner$ $\boxed{\text{using } \sum_n \tilde{\phi}_n^\dagger(l)\,\phi_n(k) = \delta(k-l)}$

$$\Rightarrow \sum_n \phi_n^\dagger\gamma_5\phi_n = \lim_{M\to\infty}\mathrm{Tr}\left\{\gamma_5\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\gamma_5\exp\left[\frac{-\not{D}^2}{M^2}\right]e^{ikx}\right\}$$

**Note: $\mathrm{Tr}\{\}$ includes trace over <u>group indices</u> (in $\not{D}$) and <u>spin indices</u>

Aside

- $\not{D}^2 = \gamma^\alpha\gamma^\beta D_\alpha D_\beta = \frac{1}{2}\left(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha + \gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)D_\alpha D_\beta = \frac{1}{2}\left(\{\gamma^\alpha,\gamma^\beta\} + [\gamma^\alpha,\gamma^\beta]\right)D_\alpha D_\beta$

$= g^{\alpha\beta}D_\alpha D_\beta + \frac{1}{2}[\gamma^\alpha,\gamma^\beta]D_\alpha D_\beta = D_\alpha D^\alpha + \frac{1}{4}\big(\underbrace{D_\alpha[\gamma^\alpha,\gamma^\beta]D_\beta + D_\beta[\gamma^\beta,\gamma^\alpha]D_\alpha}_{\text{contracted "scalar"}}\big) = D_\alpha D^\alpha + \frac{1}{4}[\gamma^\alpha,\gamma^\beta][D_\alpha,D_\beta]$

$= D_\alpha D^\alpha + \frac{1}{4}[\gamma^\alpha,\gamma^\beta]F_{\alpha\beta}$

- $e^{-ikx}\exp(\partial_\mu)e^{ikx} = e^{-ikx}\left(\sum_n \frac{1}{n!}(\partial_\mu)^n e^{ikx}\right) = e^{-ikx}\left(e^{ikx}\sum_n\frac{1}{n!}(\partial_\mu + ik_\mu)\right) = \underline{e^0\exp(\partial_\mu + ik_\mu)}$

$$\bullet \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) = \lim_{M\to\infty} Tr\left\{\gamma_5\int\frac{d^4k}{(2\pi)^4}\exp\left[-\frac{(D_\alpha+iK_\alpha)(D^\alpha+iK^\alpha)}{M^2} - \frac{[\gamma^\alpha,\gamma^\beta]F_\alpha F_\beta}{4M^2}\right]\right\}$$

$$= \lim_{M\to\infty} Tr\left\{\gamma_5\int\frac{d^4k}{(2\pi)^4}\exp\left[\frac{-D_\alpha D^\alpha}{M^2} - \frac{2iK_\alpha D^\alpha}{M^2} + \frac{K_\alpha K^\alpha}{M^2} - \frac{[\gamma^\alpha,\gamma^\beta]F_{\alpha\beta}}{4M^2}\right]\right\}$$

→ now upon rescaling $K_\alpha \to M k_\alpha$ and $K_\alpha K^\alpha = K_\alpha g^{\alpha\beta}K_\beta = -K_\alpha\delta^{\alpha\beta}K_\beta = -K^2$

$$\lim_{M\to\infty} Tr\left\{M^4\gamma_5\int\frac{d^4k}{(2\pi)^4}e^{-K^2}\exp\left[\frac{-D_\alpha D^\alpha}{M^2} - \frac{2iK_\alpha D^\alpha}{M} - \frac{[\gamma^\alpha,\gamma^\beta]F_{\alpha\beta}}{4M^2}\right]\right\}$$

- Recall that i) $Tr\{\gamma_5\} = T\{\gamma_5\gamma^\mu\gamma^\nu\} = 0$

and ii) $Tr\{\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\} = -4\varepsilon^{\mu\nu\alpha\beta}$ (euclidean)

- Now expand in factors of M and keep in mind i/ii

$$\lim_{M\to\infty} Tr\left\{M^4\gamma_5\left(\frac{1}{2!}\right)\left(-\frac{[\gamma^\alpha,\gamma^\beta]F_{\alpha\beta}}{4M^2}\right)\left(-\frac{[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}{4M^2}\right)\int\frac{d^4k}{(2\pi)^4}e^{-K^2} + \underbrace{\cdots}_{\text{zero as } M\to\infty}\right\}$$

$$= Tr\left\{\frac{\gamma_5[\gamma^\alpha,\gamma^\beta][\gamma^\mu,\gamma^\nu]F_{\alpha\beta}F_{\mu\nu}}{32}\right\}\frac{(\pi^2)}{16\pi^4} = \frac{-\varepsilon^{\alpha\beta\mu\nu}}{32\pi^2}Tr\{F_{\alpha\beta}F_{\mu\nu}\}$$

Aside

$Tr\{\gamma_5[\gamma^\alpha,\gamma^\beta][\gamma^\mu,\gamma^\nu]\} = 0$ if any indices repeat, ∴ $[\gamma^\alpha,\gamma^\beta] = 2\gamma^\alpha\gamma^\beta$ etc.

$\Rightarrow Tr\{4\gamma_5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\} = -16\,\varepsilon^{\alpha\beta\mu\nu}$

$$\bullet\ J(\beta(x)) = \exp\left[-2i\int d^4x\,\beta(x)\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x)\right]$$

$$= \exp\left[+\frac{i\varepsilon^{\alpha\beta\mu\nu}}{16\pi^2}\int d^4x\,\beta(x)\,Tr\{F_{\alpha\beta}F_{\mu\nu}\}[x]\right]$$

$$\Rightarrow Z'(A) = \int D\psi D\bar\psi\, e^{iS}\, e^{i\int d^4x\,\beta(x)(\partial_\mu J^\mu_5 - 2imP - A[A_\mu])}$$

Singlet Anomaly

- Euclidean

$$\boxed{A[A_\mu] = \frac{-i\varepsilon^{\alpha\beta\mu\nu}}{16\pi^2}Tr\{F_{\alpha\beta}F_{\mu\nu}\}}$$

or

- Minkowski

$$A[A_\mu] = \frac{\varepsilon^{\alpha\beta\mu\nu}}{16\pi^2}Tr\{F_{\alpha\beta}F_{\mu\nu}\}$$

- Regularization independence (pg 261)

–One might think the reg. procedure employed is a bit contrived, but actually the result is independent of the exact form given $f(z)$ is a smooth function subject to:

$f\left(\frac{\lambda_n^2}{M^2}\right)$; • $f(\infty) = f'(\infty) = f''(\infty) = \cdots = 0$
• $f(0) = 1$

$$\cdot \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) = \lim_{M\to\infty} \sum_n \phi_n^\dagger(x)\gamma_5 f\left(\frac{\lambda_n^2}{M^2}\right)\phi_n(x)$$

— analogous to before

$$= \lim_{M\to\infty} \int \frac{d^4k}{(2\pi)^4} Tr\left\{\gamma_5 f\left(\frac{k_\mu k^\mu}{M^2} + \frac{2ik_\mu D^\mu + D_\mu D^\mu}{M^2} + \frac{[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}{4M^2}\right)\right\}$$

• expanding $f(z)$ at point k^2/M^2

$$f(z+\varepsilon) = f\Big|_\varepsilon + f'\Big|_\varepsilon z + f''\Big|_\varepsilon \frac{z^2}{2!} + \ldots$$

with $z = \dfrac{2ik_\mu D^\mu + D_\mu D^\mu}{M^2} + \dfrac{[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}{4M^2}$, $\varepsilon = \dfrac{k_\mu k^\mu}{M^2}$ and rescale $k \to Mk$

$$= \lim_{M\to\infty} M^4 \int \frac{d^4k}{(2\pi)^4} Tr\left\{\gamma_5\left[\left(\frac{2ik_\mu D^\mu M + D_\mu D^\mu}{M^2} + \frac{[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}{4M^2}\right)^2 \frac{1}{2!} \times f''(k^2) + \ldots\right]\right\}$$

$$= \lim_{M\to\infty}\left\{\frac{M^4}{32M^4}\int\frac{d^4k}{(2\pi)^4} \underbrace{Tr\left\{\gamma_5 [\gamma^\mu,\gamma^\nu][\gamma^\alpha,\gamma^\beta]F_{\mu\nu}F_{\alpha\beta}\right\} f''(k^2)}_{\text{Only non-zero term left!}} + \underbrace{\ldots}_{0 \text{ from trace}}\right.$$

$$= Tr\left\{\frac{\gamma_5[\gamma^\mu,\gamma^\nu][\gamma^\alpha,\gamma^\beta]F_{\mu\nu}F_{\alpha\beta}}{32}\right\}\left(\frac{\pi^2}{16\pi^4}\right)\int k^2 dk^2 f''(k^2)$$

Aside

$$\int k^2 dk^2 f''(k^2) = -\int_0^\infty dk^2 f'(k^2) = -f(k^2)\Big|_0^\infty = 0 + f(0) = 1$$

$$\Rightarrow \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) = -\frac{1}{32\pi^2}\varepsilon^{\mu\nu\alpha\beta} Tr\{F_{\mu\nu}F_{\alpha\beta}\}$$

which is $\mathcal{A}[A_\mu]$!

• Next time:

① — Functional expansion + invariance of Quantum Action

③ — More applications (Fujikawa, uncertainty, Θ-vacuum, ...)

② — Realistic SM example $P_{L/R}$ with $SU(N)$