Explanation to Adler – Bardeen theorem

The usual divergence equation for ~~a~~ the axial-vector current in spinor electrodynamics is:

$$\partial^\mu j_\mu^5(x) = 2im_0 j^5(x) \quad \cancel{(*)} \text{ with}$$

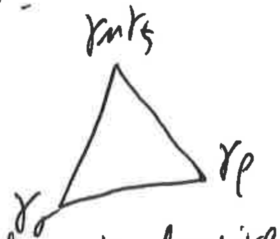
$$j_\mu^5(x) = \bar\psi(x)\gamma_\mu\gamma_5\psi(x), \quad j^5(x) = \bar\psi(x)\gamma_5\psi(x)$$

but this axial-vector current ~~fits for~~ didn't obey it, instead:

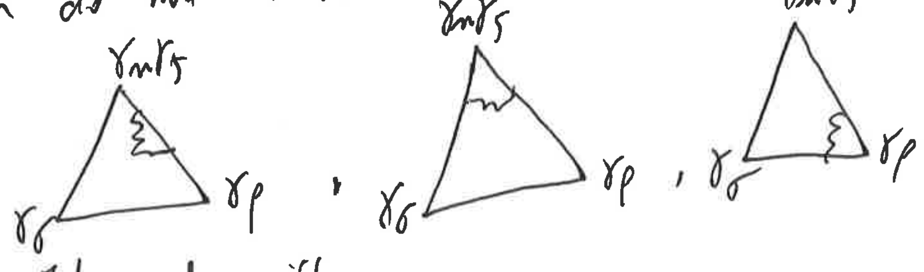
$$\partial^\mu j_\mu^5(x) = 2im_0 j^5(x) + (\alpha_0/4\pi)F^{\xi\sigma}(x)F^{\tau\rho}(x)\epsilon_{\xi\sigma\tau\rho} \quad (*)$$

with $F^{\xi\sigma}$ the electromagnetic field strength tensor.
Eq. (*) is exact. That is, the anomalous term $F^{\xi\sigma}F^{\tau\rho}\epsilon_{\xi\sigma\tau\rho}$ does not receive additional contributions from radiative corrections to triangles.

The ~~Axial~~ axial-vector triangle diagram which leads to the extra term in (*) is:



The typical second-order radiative corrections to the triangle diagram, which do not contribute to (*) are:



A. Spinor Electrodynamics

For spinor electrodynamics,

$$\mathcal{L}(x) = \bar\psi(x)(i\not\partial - m_0)\psi(x) - \tfrac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x) - e_0\bar\psi(x)\gamma_\mu\psi(x)A^\mu(x)$$

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x), \quad \not\partial \equiv \gamma^\mu\partial_\mu$$

We introduce a cutoff by modifying the usual Feynman rules.

(i) For each internal fermion line with momentum p we include a factor $i(\not p - m_0 + i\varepsilon)^{-1}$ and for each vertex a factor $-ie_0\gamma_\mu$, with m_0 and e_0 the bare mass and charge.
For each internal photon line of momentum q, we replace the usual propagator $-ig_{\mu\nu}(q^2+i\varepsilon)^{-1}$ by

$$-ig_{\mu\nu}\left(\frac{1}{q^2+i\varepsilon} - \frac{1}{q^2-\Lambda^2+i\varepsilon}\right) = \frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\,\frac{-\Lambda^2}{q^2-\Lambda^2+i\varepsilon}$$

(ii) Let $\Pi^{(2)}_{\mu\nu}(q)$ denote the two-vertex vaccum polarization loop:

$$\Pi^{(2)}_{\mu\nu}(q) = i\int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma_\mu \frac{1}{\not k - m_0 + i\varepsilon}\gamma_\nu \frac{1}{\not k + \not q - m_0 + i\varepsilon}\right]$$

It's gauge-invariant, subtracted evaluation:

$$\Pi^{(2)}_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi^{(2)}(q^2)$$

$$\Pi^{(2)}(0) = 0$$

(iii) A factor $\int d^4l/(2\pi)^4$ for each internal integration over loop variable l and a factor -1 for each fermion loop.

(iv) We use the standard, iterative renormalization procedure to fix the unrenormalized charge and mass e_0 and m_0 and the fermion wave-function renormalization Z_2 as functions of the renormalized charge and mass e and m and cutoff Λ.

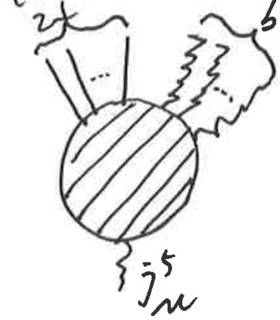
(v) We include wave-function renormalization factors $Z_2^{1/2}$ and $Z_3^{1/2}$ for each external fermion and photon line.

This simple set of rules makes all ordinary electro-dynamics matrix elements finite.

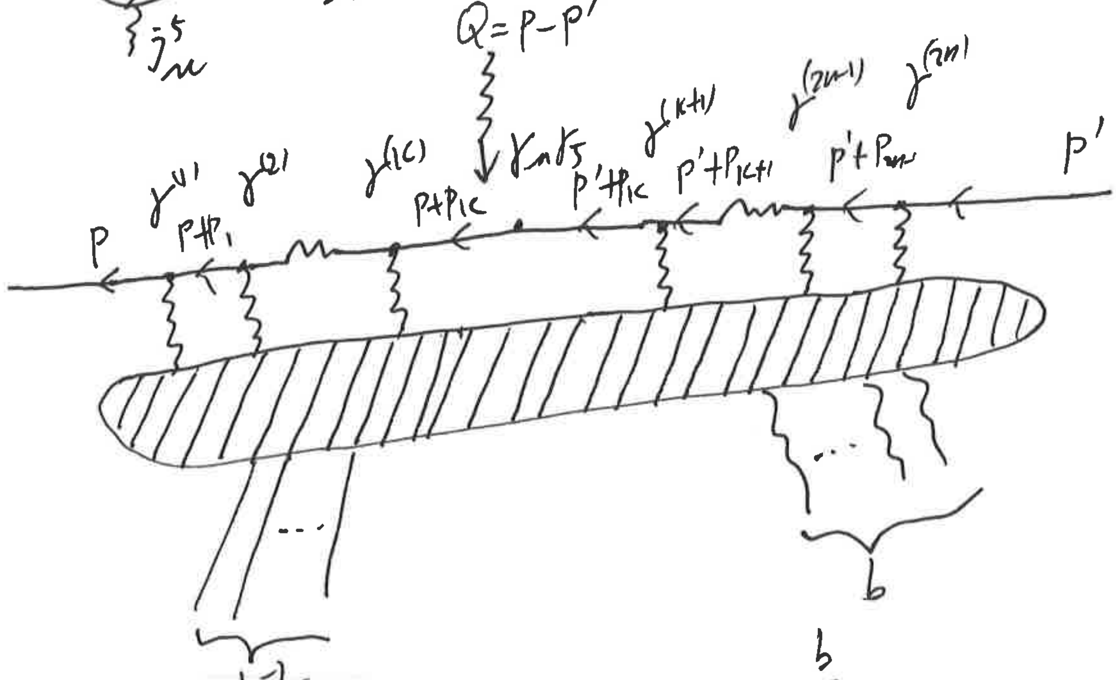
Now introduce the axial-vector and pseudoscalar currents $j_\mu^5(x)$ and $j^5(x)$, and study their properties.

First, all matrix elements of these currents are finite, ~~and~~

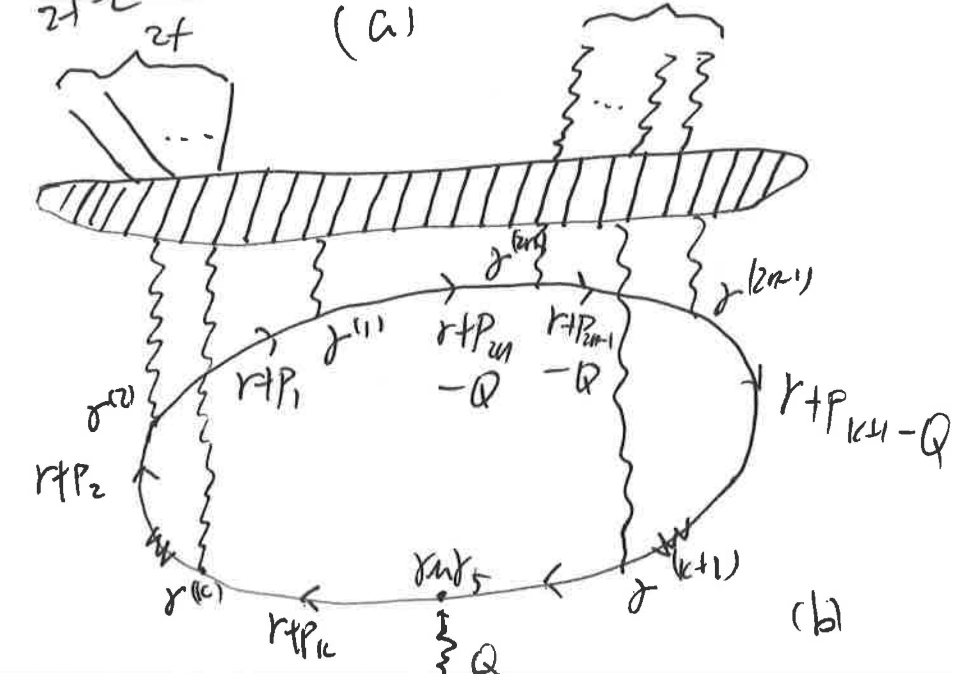
Now, we could show that (*) is exactly satisfied in the cutoff theory.



Consider an arbitrary Feynman amplitude involving j_μ^5, with 2f external fermion and b external boson lines. It's contribution could be divided into two types.



(a)



(b)

In type (a) the axial-vector vertex $\gamma_\mu\gamma_5$ is attached to one of the f fermion lines running through the diagram.

In type (b) the axial-vector vertex $\gamma_\mu\gamma_5$ is attached to an internal closed loop.

To study the divergence of the axial-vector current, we multiply the matrix element of j_μ^5 by iQ^μ. For type (a), the contribution is:

$$\sum_{k=1}^{2n-1} \prod_{j=1}^{k-1} \left[\gamma^{(j)} \frac{1}{p+p_j-m_0}\right] \gamma^{(k)} \frac{1}{p+p_k-m_0} \gamma_\mu\gamma_5 \frac{1}{p'+p_k-m_0}$$

$$\times \prod_{j=k+1}^{2n-1} \left[\gamma^{(j)} \frac{1}{p'+p_j-m_0}\right] \gamma^{(2n)} (\ldots)$$

$$Q = p - p'$$

multiplying the propagator string by iQ^μ:

$$iQ^\mu \sum_{k=1}^{2n-1} \prod_{j=1}^{k-1} \left[\gamma^{(j)} \frac{1}{p+p_j-m_0}\right] \gamma^{(k)} \frac{1}{p+p_k-m_0} \gamma_\mu\gamma_5 \frac{1}{p'+p_k-m_0}$$

$$\times \prod_{j=k+1}^{2n-1} \left[\gamma^{(j)} \frac{1}{p'+p_j-m_0}\right] \gamma^{(2n)}$$

$$= \sum_{k=1}^{2n-1} \prod_{j=1}^{k-1} \left[\gamma^{(j)} \frac{1}{p+p_j-m_0}\right] \gamma^{(k)} \frac{1}{p+p_k-m_0} 2im_0\gamma_5 \frac{1}{p'+p_k-m_0}$$

$$\times \prod_{j=k+1}^{2n-1} \left[\gamma^{(j)} \frac{1}{p'+p_j-m_0}\right] \gamma^{(2n)} - i \prod_{j=1}^{2n-1} \left[\gamma^{(j)} \frac{1}{p+p_j-m_0}\right] \gamma^{(2n)} \gamma_5 - i\gamma_5$$

$$\times \prod_{j=1}^{2n-1} \left[\gamma^{(j)} \frac{1}{p'+p_j-m_0}\right] \gamma^{(2n)}$$

The first term on the right-hand side gives $2im_0 j^5$, corresponding to replacing $\gamma_\mu\gamma_5$ by $2im_0\gamma_5$ in type (a). The two remaining terms are the usual 'surface terms' and will vanish.

For type (b), it's contribution written as

$$L(Q;\gamma_\mu\gamma_5;P_1,\cdots,P_{2n-1})(\cdots)$$

$$L(Q;T;P_1,\cdots,P_{2n-1}) = \int d^4r\, \mathrm{Tr}\left\{ \sum_{k=1}^{2n} \prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+P_j-m_0}\right]\gamma^{(k)}\frac{1}{r+P_k-m_0} T \right.$$

$$\left. \times \frac{1}{r+P_k-Q-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+P_j-Q-m_0}\right]\right\}$$

$$L(Q;iQ^\mu\gamma_\mu\gamma_5;P_1,\cdots,P_{2n-1}) = L(Q;2im_0\gamma_5;P_1,\cdots,P_{2n-1})$$

$$+ i\int d^4r\,\mathrm{Tr}\left\{\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+P_j-m_0}\right]\right.$$

$$\left. -\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+P_j-Q-m_0}\right]\right\}$$

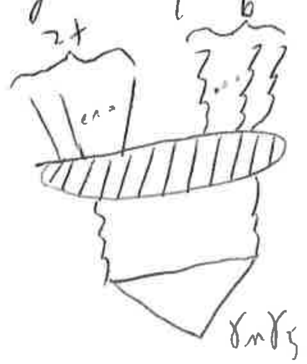
For loops $n\geq 2$, the residue integral in it is sufficiently convergent to make ~~the~~ $r\to r+Q$, causing the two terms cancel.

So, for $n\geq 2$,

$$L(Q;iQ^\mu\gamma_\mu\gamma_5;P_1,\cdots,P_{2n-1}) = L(Q;2im_0\gamma_5;P_1,\cdots,P_{2n-1})$$

Then the type (b) pieces containing loops with $n\geq 2$ all agree with the usual divergence equation $\partial^\mu j_\mu^5(x) = 2im_0 j^5(x)$.

For $n=1$, it's like:



When the triangle is defined to be gauge-invariant with respect to the vector indices, it doesn't satisfy this eq. for the axial-vector index divergence. Instead, there is a well-defined extra term left over. It could be shown that the extra term is to add to the normal axial-vector divergence equation the term

$$(\alpha_0/4\pi)\,[F^{\xi\sigma}(x)+F^{R\xi\sigma}(x)]\,[F^{\tau\rho}(x)+F^{R\tau\rho}(x)]\,\epsilon_{\xi\sigma\tau\rho}$$

So, the diagrammatic analysis at here shows that the axial-vector divergence equation in the regulated field theory is

$$\partial^{\mu} j_{\mu}^{5}(x) = 2 i m_0 j^{5}(x) + (\alpha_0/4\pi)\, F^{\xi\sigma}(x) F^{\tau\rho}(x)\,\epsilon_{\xi\sigma\tau\rho} + (\alpha_0/4\pi)\,[F^{\xi\sigma}(x)F^{R\tau\rho}(x) + F^{R\xi\sigma}(x)F^{\tau\rho}(x) + F^{R\xi\sigma}(x)F^{R\tau\rho}(x)]\,\epsilon_{\xi\sigma\tau\rho}$$

Which is identical to (*), apart from the term involving F^R which arise from the explicit inclusion of a regulator field.

Also, the second-order radiative corrections to the $\gamma_5-\gamma_\sigma-\gamma_\rho$ triangle in spinor electrodynamics, by calculation, is just zero.