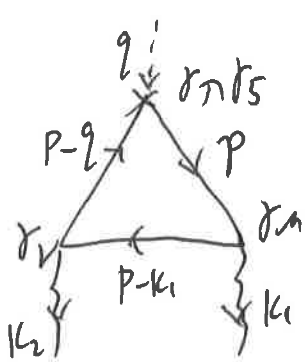
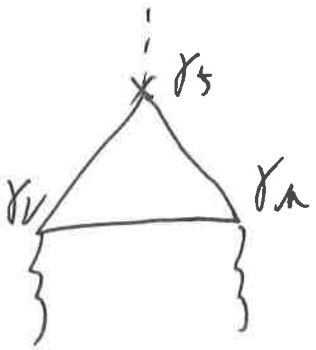
1.

$T_{\mu\nu\lambda}$:



$+$

$T_{\mu\nu}$:



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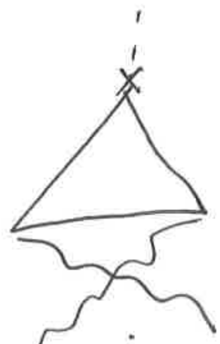

$$T_{\mu\nu\lambda} = i\int \frac{d^4p}{(2\pi)^4}(-)\,\mathrm{tr}\,\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{\not p-\not q-m}\gamma_\nu\frac{i}{\not p-\not k_1-m}\gamma_\mu + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu\leftrightarrow\nu\end{pmatrix}$$

$$T_{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4}(-)\,\mathrm{tr}\,\frac{i}{\not p - m}\gamma_5\frac{i}{\not p-\not q-m}\gamma_\nu\frac{i}{\not p-\not k_1-m}\gamma_\mu + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu\leftrightarrow\nu\end{pmatrix}$$

with $q = k_1 + k_2$

Axial Ward identity (AWI):

Use the identity:

$$\not q\gamma_5 = \gamma_5(\not p-\not q-m) + (\not p-m)\gamma_5 + 2m\gamma_5$$

then we could get: $q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + R^1_{\mu\nu} + R^2_{\mu\nu}$

where: $$R^1_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} tr\Big[\frac{1}{\not p - \not k_2 - m} \gamma_5 \gamma_\nu \frac{1}{\not p - \not q - m} \gamma_\mu - \frac{1}{\not p - m} \gamma_5 \gamma_\nu \frac{1}{\not p - \not k_1 - m} \gamma_\mu \Big]$$

$$R^2_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} tr\Big[\frac{1}{\not p - \not k_1 - m} \gamma_5 \gamma_\mu \frac{1}{\not p - \not q - m} \gamma_\nu - \frac{1}{\not p - m} \gamma_5 \gamma_\mu \frac{1}{\not p - \not k_2 - m} \gamma_\nu \Big]$$

If $R^1_{\mu\nu}, R^2_{\mu\nu} \to 0$, then the AWI is satisfied. This happens if we make the shift:

$$p \to p + k_2, \quad p \to p + k_1$$

However, the integrals are linearly divergent. So ~~we~~ the two shifts are not allowed, and

$$R^1_{\mu\nu}, R^2_{\mu\nu} \neq 0$$

$\therefore$ AWI becomes anomalous.

2. A little bit on linear divergent integrals:

For example: $\Delta(a) = \int_{-\infty}^{\infty} dx\,[f(x+a) - f(x)] = \int_{-\infty}^{\infty} dx\,[a f'(x) + \frac{a^2}{2!} f''(x) + \cdots]$

$$= a[f(\infty) - f(-\infty)] + \frac{a^2}{2!}[f'(\infty) - f'(-\infty)] + \cdots$$

If $\int_{-\infty}^{\infty} dx\, f(x)$ converges, then $f(\pm\infty) = f'(\pm\infty) = \cdots = 0$, so $\Delta(a) = 0$.

In the case we could shift $x \to x - a$.

If, however, this integral is linear divergent, then $f(\pm\infty) \neq 0$, $f'(\pm\infty) = f''(\pm\infty) = \cdots = 0$, and

$$\Delta(a) = a[f(\infty) - f(-\infty)] \neq 0$$

So we can't make the shift $x \to x - a$.

For n dimensions:

$$\Delta(a) = \int d^n x\, [f(x+a) - f(x)]$$

Gauss theorem

$$= \int d^n x\, [a^\mu \partial_\mu f(x) + a^\mu a^\nu \partial_\mu \partial_\nu f(x) + \ldots]$$

$$= a^\mu \lim_{R\to\infty} \frac{R_\mu}{R} S^{n-1}(R) f(R)$$

$S^{n-1}(R)$ is the surface of an $(n-1)$-dimensional sphere.

For $n=4$, $\quad S^3(R) = 2\pi^2 R^3$

For $R^{1,2}_{\mu\nu}$, they have the form:

$$R^1_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} [f(p-k_2) - f(p)]$$

$$R^2_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} [f(p-k_1) - f(p)]$$

So: ~~For~~ $R^1_{\mu\nu} = i2\pi^2(-k_2^\lambda) \lim_{p\to\infty} p_\lambda p^2 \frac{tr(\not p + m)\gamma_5 \gamma_\nu (\not p - \not k_1 + m)\gamma_\mu}{(2\pi)^4 [p^2 - m^2][(p-k_1)^2 - m^2]}$

$$= \frac{1}{2\pi^2} \varepsilon_{\beta\nu\alpha\mu} k_1^\alpha k_2^\lambda \lim_{p\to\infty} \frac{p_\lambda p^\beta}{p^2}$$

$$= -\frac{1}{8\pi^2} \varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

$k_1 \leftrightarrow k_2$, $\mu \leftrightarrow \nu$, we get:

$$R^2_{\mu\nu} = -\frac{1}{8\pi^2} \varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

$\therefore$ The AWI is:

$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} - \frac{1}{4\pi^2} \varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

3. Ambiguity of the amplitude:

Do the shift $\quad p \to p + a$, with

$$a = \alpha k_1 + (\alpha - \beta) k_2$$

and $\Delta_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) = -\int \frac{d^4p}{(2\pi)^4}$

$$\cdot\Big[tr \frac{1}{\not p+\not a-m}\gamma_\lambda\gamma_5 \frac{1}{\not p+\not a-\not q-m}\gamma_\nu \frac{1}{\not p+\not a-\not k_1-m}\gamma_\mu$$

$$-tr\frac{1}{\not p-m}\gamma_\lambda\gamma_5\frac{1}{\not p-\not q-m}\gamma_\nu\frac{1}{\not p-\not k_1-m}\gamma_\mu\Big]+$$

$$\begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu\end{pmatrix} =: \Delta^1_{\mu\nu\lambda}+\Delta^2_{\mu\nu\lambda}$$

In the limit $p\to\infty$,

$$\Delta^1_{\mu\nu\lambda} = -\frac{1}{8\pi^2}\varepsilon_{\mu\nu\lambda\alpha}a^\alpha$$

For $k_1\leftrightarrow k_2$ and $\mu\leftrightarrow\nu \Rightarrow \Delta^2_{\mu\nu\lambda}$

$$\therefore \Delta_{\mu\nu\lambda} = \Delta^1_{\mu\nu\lambda}+\Delta^2_{\mu\nu\lambda} = -\frac{\beta}{8\pi^2}\varepsilon_{\mu\nu\lambda\alpha}(k_1^\alpha-k_2^\alpha)$$

$$T_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(0)+\Delta_{\mu\nu\lambda}(a)$$

$\therefore$ The AWI ($\beta\neq1$):

$$q^\lambda T_{\mu\nu\lambda}(\beta) = 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta$$

4. Vector Ward identity (VWI):

$$k_1^\mu T_{\mu\nu\lambda} = -\int\frac{d^4p}{(2\pi)^4} tr\Big[\frac{1}{\not p-m}\gamma_\lambda\gamma_5\frac{1}{\not p-\not q-m}\gamma_\nu\frac{1}{\not p-\not k_1-m}\not k_1$$

$$+\frac{1}{\not p-m}\gamma_\lambda\gamma_5\frac{1}{\not p-\not q-m}\not k_1\frac{1}{\not p-\not k_2-m}\gamma_\nu\Big]$$

$$= -\int\frac{d^4p}{(2\pi)^4} tr\Big[\gamma_\lambda\gamma_5\frac{1}{\not p-\not q-m}\gamma_\nu\frac{1}{\not p-\not k_1-m}$$

$$-\gamma_\lambda\gamma_5\frac{1}{\not p-\not k_2-m}\gamma_\nu\frac{1}{\not p-m}\Big]$$

$$=: -\int \frac{d^4p}{(2\pi)^4}\left[f(p-k_1)-f(p)\right]$$

From $k_1^\mu T_{\mu\nu\lambda}(0)=\frac{1}{8\pi^2}\varepsilon_{\beta\nu\alpha\lambda}k_1^\alpha k_2^\beta$ we could get the anomalous VWI $(\beta\neq -1)$:

$$k_1^\mu T_{\mu\nu\lambda}(\beta)=\frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\alpha\beta}k_1^\alpha k_2^\beta$$

$\therefore$ Obviously, there is no value of β such that both the VWI and the AWI can be fulfilled.

For example:

i) $\beta=-1$, then the VWI

$$k_1^\mu T_{\mu\nu\lambda}=0$$

is satisfied. But the AWI is anomalous

$$q^\lambda T_{\mu\nu\lambda}=2mT_{\mu\nu}+A_{\mu\nu}$$

with $A_{\mu\nu}=-\frac{1}{2\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta$

ii) $\beta=1$, AWI is fulfilled

$$q^\lambda T_{\mu\nu\lambda}=2mT_{\mu\nu}$$

but the VWI is anomalous

$$k_1^\mu T_{\mu\nu\lambda}=-\frac{1}{2}A_{\nu\lambda}$$

iii) $\beta=1/3$, the anomalous is equally distributed in the VWI and AWI.

$$k_1^\mu T_{\mu\nu\lambda}=-\frac{1}{3}A_{\nu\lambda}$$

$$q^\lambda T_{\mu\nu\lambda}=2mT_{\mu\nu}+\frac{1}{3}A_{\mu\nu}$$

Adler–Bardeen theorem: The full structure of the chiral anomaly is given by the triangle graph.