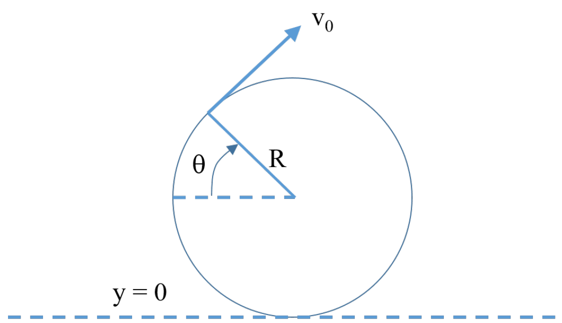
Classical Mechanics

1. Consider a wheel of radius R which is orientated vertically and is spinning about its center, such that the speed of a particle at the outer edge of the wheel is v_0. Imagine that particles of mud are being thrown up by the spinning wheel, and that they are launched from the wheel edge with an initial speed of v_0, at a tangent to the wheel, as shown in the diagram below. Assume that a particle can be launched at any angle as the wheel rotates.



   (a) Find the launch angle $\theta_{\mathrm{m}ax}$ relative to the horizontal axis which causes the thrown particle to reach the greatest height (relative to the bottom of the wheel, $y = 0$ in the diagram).

   (b) Show that the greatest height (relative to the bottom of the wheel) that a launched particle can reach is given by:
$$R + \frac{v_0^2}{2g} + \frac{gR^2}{2v_0^2}$$

   (c) What condition relating the radius R to the speed v_0 is required for this solution to be valid?

2. A block of mass m, attached to a spring of spring constant k, is moving in one dimension on a horizontal surface. The coefficient friction between the block and the surface is $\mu = 1/5$, taken to be the same for static and kinetic friction. The mass is at $x = 0$ in equilibrium (spring unstretched).

   a) Suppose the spring is extended to $x = A$ (say, to the right) and the mass is released (from rest). Show that the block does not move if A is less than some critical distance x_m, and find x_m.

b) Suppose $A = 8x_m$ when the mass is released.

Find i) the position of the mass x, (ii) energy lost due to friction, and (iii) the time taken (period) at the end of one cycle (where mass goes to the left and then comes back again to the right).

3. Consider one dimensional motion of a particle with mass m. This particle is moving in the direction of the positive x-axis under a velocity-dependent drag force $F = -\frac{v^p}{A}$, where $p > 0$ and $A > 0$ are constants with the appropriate units. At $t = 0$, the particle is at the origin and has a velocity $v = v_0 > 0$.

   (a) Using Newton's 2nd Law, write the differential equation of motion for the particle.

   (b) Find a condition on p for the particle to stop within a finite time (time $t < \infty$, when the particle stops).

   (c) Find a condition on p for the particle to stop within a finite distance (the position of the particle $x < \infty$, when it stops).

4. Consider a simple pendulum (length ℓ, mass m) whose support point is moving vertically upward with a constant acceleration a.

   (a) Construct the Lagrangian for this system using the oscillation angle θ (relative to the vertical axis) as the generalized coordinate.

   (b) Use Lagrange's equation to find the equation of motion for θ, and show that the period of small oscillations for this rising pendulum is given by $2\pi\sqrt{\ell/(g+a)}$. Comment briefly on why this should be the expected result?

5. The center of a long frictionless rod is pivoted at the origin, and the rod is forced to rotate in a horizontal plane with constant angular velocity ω.

   (a) Write down the Lagrangian for a bead of mass m threaded on the rod, using r as your generalized coordinate, where r, ϕ are the polar coordinates of the bead. Note that ϕ is not an independent variable since it is fixed by the rotation of the rod to be $\phi = \omega t$.

   (b) Solve Lagrange's equation for $r(t)$ and express a general solution with two constants of integration.

   (c) What happens if the bead is initially at rest at the origin?

   (d) If the bead is released from any point $r_0 > 0$, show that $r(t)$ eventually grows exponentially.

6. A particle of mass m is moving under a central force $f(r)$.

   Prove that a) energy and b) angular momentum vector ($\mathbf{L}$) are constants of the motion (independent of time).

   c) Now consider an orbit in the x-y plane, so that the angular momentum (magnitude L) is in the z direction. The orbit is described by $r = ae^{b\theta}$, in polar coordinates. Starting from the expression for the angular momentum L_z in polar coordinates, find the angle as a function of time, $\theta(t)$, for this orbit, in terms of the constants a, b, m & L, given that $\theta = 0$ at $t = 0$.

Electromagnetism

1: Radiation. The retarded potentials of electromagnetism in the Lorentz gauge $c^{-1}\partial_t\phi + \nabla \cdot \mathbf{A} = 0$ read as

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c}\int d^3r' \, \frac{\mathbf{j}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}, \qquad \phi(\mathbf{r}, t) = \int d^3r' \, \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}. \tag{1}$$

The charge and current densities ρ and $\mathbf{j}$ are related by the continuity equation $\partial_t\rho + \nabla \cdot \mathbf{j} = 0$. The aim of this problem is to find the leading contribution to the radiation fields in the near field regime. The source is located close to the origin, with typical size d. The time dependence of the charge and current densities is given by $\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) \exp(-i\omega t)$, and $\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_0(\mathbf{r}) \exp(-i\omega t)$. In the following, you may calculate with complex fields ρ, $\mathbf{j}$, $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$. Note that the physical fields corresponds to the real or imaginary parts.

(a) Show that the leading contribution to the vector potential in the limit $d \ll r$ can be written as $\mathbf{A}(\mathbf{r}, t) = -ik\mathbf{p}(t)\mathrm{e}^{ikr}/r$, where $k = \omega/c$ and $\mathbf{p}(t) = \int d^3r \; \mathbf{r}\rho(\mathbf{r}, t)$. HINT: Use the relation $\int d^3r \; j_i = \int d^3r \; \nabla r_i \cdot \mathbf{j}$ and the continuity equation to introduce the dipole moment. [10pts]

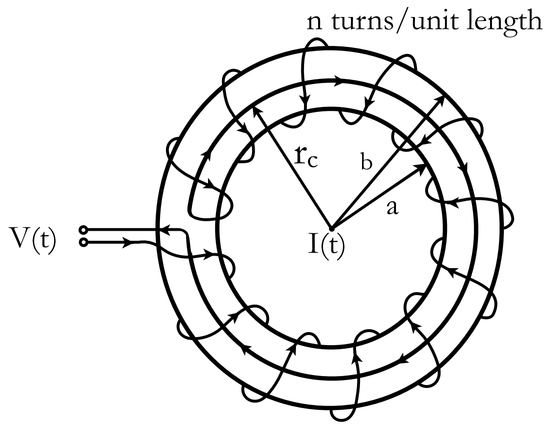
(b) Find the magnetic and electric fields in the near field approximation $r \ll \lambda$, where λ is the wavelength. You may use the result for $\mathbf{A}$ stated in (a) as a starting point. HINT: You can find $\mathbf{E}$ from $\mathbf{B}$ with the help of the Faraday-Maxwell equation in the absence of currents. [10pts]

2: Rogowksi coil.

A Rogowski coil is constructed of a soft iron torus (with $\mu \gg 1$) of mean radius r_c, with circular cross section of diameter $d = b - a \ll r_c$. It is wound along its entire diameter with thin wire with a number of turns per unit length n. A wire with current $I(t) = I_0 \cos(\omega t)$ is fed through the hole in the center of the torus.

(a) Using the quasistatic approximation, find $\mathbf{H}(t)$ inside the torus. [8pts]

(b) Find the induced EMF, $V(t)$, across the winding of the torus. [12pts]



3: Electric and magnetic fields in matter. Consider a metallic charged wire with circular cross section of radius a and constant linear charge density λ. This wire is coated by a linear dielectric of permittivity ε so that the dielectric material fills the space up to a radius b.

(a) Find the electric field in the regions I. $\rho < a$, II. $a < \rho < b$, III. $\rho > b$, where ρ is the distance from the central axis. [10pts]

(b) Find (i) the polarization $\mathbf{P}$ inside the dielectric and from that (ii) the charge density ρ_b of bound charges inside the dielectric and (iii) the surface charges σ_{outer} and σ_{inner} for both surfaces of the dielectric. [10pts]

4: Gauss Law. A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2}(a \leq r \leq b). \tag{2}$$

Find the electric field in the three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$. Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

5: Electromagnetic Waves. Suppose

$$\mathbf{E}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]\hat{\phi}, \tag{3}$$

with $\omega/k = c$. This is, incidently, the simplest possible spherical wave. For notation convenience, let $u \equiv (kr - \omega t)$ in your calculations.

(a) Show that $\mathbf{E}$ obeys all four Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average $\mathbf{S}$ over a full cycle to get the intensity vector $\mathbf{I}$. (Does it point in the expected direction? Does it fall off like r^{-1}, as it should?)

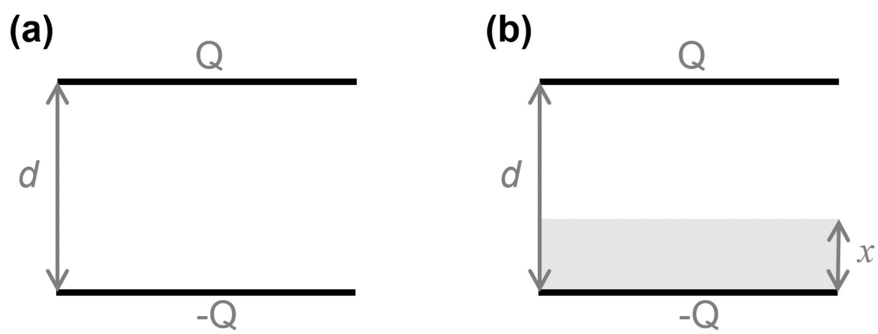
(c) Integrate $\mathbf{I} \cdot d\mathbf{a}$ over a spherical surface to determine the total power radiated. [*Answer* : $4\pi A^2/3\mu_0 c$]

6: Electrostatics: This problem contains two steps:

(a) Suppose you have a parallel plates capacitor with charge Q with the plates distant d from each other (see Fig.2(a)). The area of the plates is A and there is vacuum between them, whose permittivity constant is ε_0. Calculate the energy store in the capacitor and then calculate the work done by the electrostatic forces if you approximate the plates by an amount δ.

(b) Now, supposed that you partially filled the capacitor with a liquid linear dielectric (with electric susceptibility χ_e) by pouring it until it reaches the desired height x, as show in Fig.2(b). Calculate the energy stored by this new capacitor.

(c) Finally, you pour a little bit more dielectric in the capacitor until it reaches the height $x + \delta$. As you do this, what is now the work done by the electrostatic forces in terms of that work found for the empty capacitor?



Quantum Mechanics

1. A non-relativistic particle of mass m moves in three dimensions under the influence of a central attractive force of constant magnitude κ. The corresponding potential energy is

$$V(\vec{r}) = \kappa r$$

Use the variational principle to obtain an estimate of the ground state energy in terms of κ, m, and any physical and mathematical constants.

The time–independent Schrodinger equation in spherical coordinates is

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right] + V(\vec{r})\psi = E\psi$$

Take as given

$$\int_0^\infty dx x^n e^{-x} = n! \qquad \int_0^\infty x^{2n} e^{-x^2} = \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}} \qquad \int_0^\infty x^{2n+1} e^{-x^2} = \frac{n!}{2}$$

where $n = 0, 1, 2, 3....$ and $m!! = m \cdot (m-2) \cdot (m-4) \cdots 1$ for m odd.

2. Consider an unbound particle of mass m with a binary internal degree of freedom propagating with energy E in one space dimension under the influence of the potential

$$V(x) = \begin{cases} V_0 \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} & x > 0 \\ 0 & x < 0 \end{cases}$$

where $V_0 > 0$.

   (a) For each region, $x < 0$ and $x > 0$, find a complete set of independent energy eigenfunctions in terms of m, V_0, E, and any physical and mathematical constants. (Boundary conditions can be ignored for this part.)

   (b) Suppose that the particle is prepared as a state initially propagating in the $+x$ direction from $-\infty$ with the internal degree of freedom in the state corresponding to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find the probability, in terms of m, V_0, E, and any physical and mathematical constants, that the particle is reflected from $x = 0$ with the internal degree of freedom in the state corresponding to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

3. Say the Hamiltonian for a spinning particle is given by

$$H = \mathcal{E}_0 \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} , \qquad \mathcal{E}_0 \text{ is real} ,$$

while the spin matrices are

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Is H hermitean?

(b) Can the energy be measured simultaneously with the spin in either the x, y or z-direction?

(c) What energy spectrum follows from H?

(d) If S_z is measured to be $\hbar/2$ at time $t = 0$, what is the likelihood of measuring S_z to be $\hbar/2$ for $t > 0$?

(e) If *instead*, S_y is measured to be $\hbar/2$ at $t = 0$, what is the likelihood of measuring S_y to be $\hbar/2$ for $t > 0$?

4. In a 1-dimensional quantum mechanical system, a particle with mass m is trapped in a potential well. The eigenfunction of a bound state with energy $E = -\frac{\hbar^2}{2mL^2}$ is given by

$$\Psi(x) = \begin{cases} A\, x\, e^{-\frac{x}{L}} & (0 \leq x) \\ 0 & (x < 0) \end{cases}$$

Here, A is a constant, and $L > 0$ is a length. In solving the following problems, one may use

$$\int_0^\infty dx\; x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

for a real and positive parameters a and the integer $n = 1, 2, 3, \cdots$.

(a) Fix the normalization constant A to satisfy $\int_{-\infty}^{\infty} dx \Psi(x)^* \Psi(x) = 1$ (up to a complex phase).

(b) Calculate the expectation value of the position $\langle x \rangle$.

(c) Calculate the expectation value of the squared momentum $\langle p^2 \rangle$.

(d) Find the potential of the system $V(x)$ for $x \geq 0$.

5. We consider the 1-dimensional harmonic oscillator with the potential

$$V(x) = \frac{1}{2} m \omega_c^2 x^2 .$$

We introduce the annihilation operator (a) and the creation operator ($a^\dagger$) defined as

$$a = \sqrt{\frac{\hbar}{2m\omega_c}} \left(\frac{d}{dx} + \frac{m\omega_c}{\hbar} x \right) , \quad a^\dagger \;=\; \sqrt{\frac{\hbar}{2m\omega_c}} \left(-\frac{d}{dx} + \frac{m\omega_c}{\hbar} x \right) .$$

(a) Verify that $[a, a^\dagger] = 1$ and the Hamiltonian is expressed as $H = \hbar\omega_c \left(aa^\dagger - \frac{1}{2}\right)$.

(b) Verify $[H, a] = -\hbar\omega_c a$ and $[H, a^\dagger] = +\hbar\omega_c a^\dagger$.

(c) Derive the ground state eigenfunction u_0 from $au_0 = 0$. Use the result to find the eigenfunction for the first excited state. You need not normalize the eigenfunctions.

(d) Suppose we have an eigenfunction v_0 which satisfies $a^\dagger v_0 = 0$. Show that this eigenfunction satisfies $Hv_0 = -\frac{1}{2}\hbar\omega_c v_0$, and find an explicit formula for v_0 as a function of x. You do not need to normalize the wavefunction.

(e) From the mathematical point of view, v_0 is a solution to the Schrödinger equation and we may adopt it as an eigenstate. However, in the view point of physics, we exclude v_0 from our discussion about the harmonic oscillator. Explain the reason.

6. In a 3-dimensional system, a particle of mass m is trapped in an infinite potential well (box),

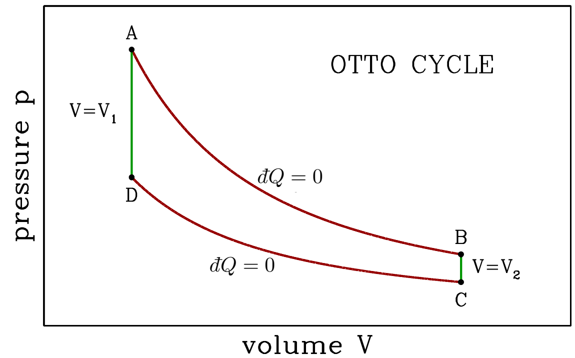
$$V(x, y, z) = \begin{cases} 0 & (0 \le x \le L,\ 0 \le y \le 2L,\ 0 \le z \le 3L) \\ \infty & (x, y, z < 0 \text{ and } L < x,\ 2L < y,\ 3L < z) \end{cases}$$

(a) Write the solution of the time-independent Schrödinger equation in the region inside the well.

(b) Determine the energy spectrum of the system.

1: Ideal gas and Otto cycle

The Otto cycle consists of two adiabats and two isochores as illustrated in the figure.



Assume that the working substance is an ideal gas. The ideal gas is characterized by the gas law $pV = \nu RT$, $\nu = N/N_A$, where N_A is the Avogadro number, and the energy $E(N, V, T) = \nu\varepsilon(T)$.

(a) Assume that the particle number is constant. The specific heat at constant volume and the specific heat at constant pressure are defined as

$$c_V = \frac{1}{\nu}\left.\frac{đQ}{dT}\right|_V, \qquad c_P = \frac{1}{\nu}\left.\frac{đQ}{dT}\right|_p \tag{1}$$

Show that they are related as $c_p = c_V + R$ [15 pts].

(b) For the ideal gas, c_V and c_p are temperature-independent. Show that for an adiabatic transformation ($đQ = 0$) the following relation holds [15 pts]:

$$pV^\gamma = \text{const.}, \qquad \gamma = \frac{c_p}{c_V}. \tag{2}$$

(c) Determine the heat transfer Q_{ij} to the system, the work W_{ij} performed by the system and the change of the energy of the system $\Delta E_{ij} = E_j - E_i$ for each process $ij \in \{AB, BC, CD, DA\}$ during the Otto-cycle. Let W denote the total work performed by the system during the cycle. Show that the efficiency $\eta \equiv W/Q_{DA}$ is given as [20 pts]:

$$\eta = 1 - \left(\frac{V_1}{V_2}\right)^{\gamma-1}. \tag{3}$$

2: Blackbody radiation

Thermodynamics of blackbody radiation is described in terms of a gas of photons confined in a volume V and in thermal equilibrium at a temperature T. The energy of a photon of wave vector $\vec{k}$ is equal to $\hbar ck$, where k is the magnitude of $\vec{k}$.

(a) Use the appropriate distribution function to express the average energy per unit volume U/V as an integral over $\vec{k}$ and show by power counting or otherwise that it is proportional to T^4. Hint: You do *not* need to evaluate the integral [17 pts].

(b) The radiant energy flux (energy emitted per unit area per unit time through a hole in the wall) of a black body is given by $J = \sigma T^4$ where σ is a constant. Notation: Let R_S be the radius of the sun, R_{ES} the earth-sun distance, R_E the radius of the earth, and T_S the surface temperature of the sun.

   i. Find an expression for the total energy radiated by the sun (treated as a black body) per unit time in terms of R_S and T_S [15 pts].

   ii. Assuming that the surface of the earth (treated as a black body) radiates as much energy as it receives from the sun, show that the surface temperature of the earth is given by $T_S(R_S/2R_{ES})^{1/2}$ [18 pts].

3: Ensemble of pressures

The Ensemble of Pressures: Imagine that you bring a system into contact with a reservoir of heat and work (or Thermal and volume reservoirs). The wall connecting the system and the reservoir is diathermal and can also move (consider that $\tau = k_b T$).

(a) Show that the ratio between the probability $P(V_1, \varepsilon_1)$ to find the system in a state with energy ε_1 and volume V_1 and the probability $P(V_2, \varepsilon_2)$ to find the system in a state with energy ε_2 and volume V_2 is given by [19 pts]:

$$\frac{P(V_1, \varepsilon_1)}{P(V_2, \varepsilon_2)} = \frac{exp\left[\frac{-V_1 p - \varepsilon_1}{\tau}\right]}{exp\left[\frac{-V_2 p - \varepsilon_2}{\tau}\right]}. \tag{4}$$

(b) Show that the partition function can be written as $z = \sum_v \sum_\varepsilon exp\left[\frac{-Vp-\varepsilon}{\tau}\right]$, where the summation goes over all volumes and energies [6 pts].

(c) Find expressions for U $= <\varepsilon>$ (average internal energy) and $<V>$ (average volume) in terms of derivatives of the partition function [19 pts].

(d) Find an expression for the compressibility at $\tau =$ const, $\kappa_\tau = -\frac{1}{V}\frac{\partial V}{\partial p}$, in terms of derivatives of the partition function [6 pts].