Motivation.

We learned/Reviewed some aspects of SUSY. Now the question is what is an index what is the problem at hand and why?

The Theorem: in simple words The Atiyah-Singer index states the topological index of a ~~differentiable~~ differential operator is equal to its analytic index.

Here we consider elliptic differential op.

Diff. op.: is a mapping (linear) between smooth sections of two vector bundles over a manifold. locally this looks like a linear combination of partial derivatives.

Elliptic: ker(D), coker(D) are finite-dimensional

That is $Df = g$

has finitely many linearly independent solutions for f ; a given g.

And only finitely many restrictions for what g's ~~give~~ have solutions.

ker − coker = ind. →

Problem! This means we potentially need to tabulate ~~an~~ ~~infinite~~ the set of all possible solutions.

This is hopeless!

An outline of what follows:

- A toy index Theorem to get our feet wet. 9/15
- A look at the Playground involved (i.e. the topology)
- Def of Dirac operator on this playground. 9/22
- Analytic index of D.
- Statement of The index Theorem in General and What its reduction to a spin complex looks like. 9/29
- Proof.

Along the way we will develop any other needed materials as we go.

- its worth emphasizing how deeply the interplay between mathematics & Physics is. A Physically Relavant calculation can answer a fundamentally mathematical question. What will see is $\dim\ker D - \dim\ker D^{+}$ is The difference of + & - chirality zero modes.

Simple example:

let V & W be vector spaces. & $f: V \to W$ a linear map. The *image* of f is $f(V) \subseteq W$. The *kernel* is $\ker f = \{v \in V \mid f(v) = 0\}$

So we have

$\mathrm{Im} f = \{ y \mid y = f(v),\ v \in V \}$, $\mathrm{Im} f \subset W$

$\mathrm{Ker} f = \{ v \mid f(v) = 0 \}$

Theorem: if $f: V \to W$ is linear

$$\dim V = \dim(\ker f) + \dim(\mathrm{im} f)$$

Proof involves considering basis $\{g_1 \cdots g_l\} \cdot \ker f$
$\{h_1 \cdots h_s\} = \mathrm{im} f$

and showing this forms a basis for V.
- if they form a basis then $\dim V$ is the # of elements in the basis for finite V.

metric - is a vector space isomorphism

$g: V \to V^*$ $\quad g \in GL(m, \mathbb{R})$, $V(m, \mathbb{R})$

$g: v^i \to g_{ij} v^j$

This defines an inner product

$$g(v_1, v_2) = g_{ij} v_1^i v_2^j$$

let W be a seperate vector space with $G: W \to W^*$ then with a map $f: V \to W$ the adjoint is defined so that

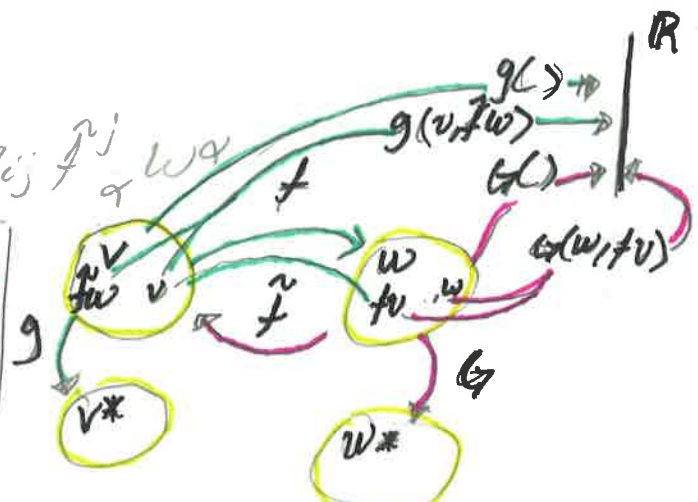
$$G(w, fv) = g(v, \tilde{f} w)$$

$$w^\alpha G_{\alpha\beta} f^\beta{}_i v^i = v^i g_{ij} \tilde{f}^j{}_\alpha w^\alpha$$

Theorem: let V & W be finite dim vector spaces & $f: V \to W$ be linear then

$$\dim(\ker f) - \dim(\ker \tilde{f}) = \dim V - \dim W$$

RHS does not depend on f!



This is a toy version of the index theorem

Opening example:

Suppose i have vector spaces V & W with $\dim V = 3$ $\dim W = 2$.

let g & G be vector space isomorphisms for V & W respectfully i.e. $g: V \to V^*$ & $g: V \times V^* \to \mathbb{R}$

$G: W \to W^*$ $G: W \times W^* \to \mathbb{R}$

Then I can tell you if $f: V \to W$ is a linear map that $\dim\ker f - \dim\ker \tilde{f} = 1$. Give intro now

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now return to example after definitions.

Supose $f: V \to W$ is given $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \begin{pmatrix} 2x + y \\ y + 2z \end{pmatrix}$

Then let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in W$ define the adjoint $\tilde{f}$ so that

$$g(\vec{x}, \tilde{f}u) = G(f\vec{x}, u)$$

if $\tilde{f}: W \to V$ as $\tilde{f}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2u_1 \\ u_1 + u_2 \\ 2u_2 \end{pmatrix}$

$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then $\ker(f) \equiv \{v \in V \mid f(v) = 0\}$

$\Rightarrow$ $\begin{pmatrix} 2x + y \\ y + 2z \end{pmatrix} = \vec{0}$ $\quad x = -y/2$

$z = -y/2$

$\rightarrow$ $y\begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix}$ $\dim\ker f = 1$

$\ker(\tilde{f}) \rightarrow \begin{pmatrix} 2u_1 \\ u_1 + u_2 \\ 2u_2 \end{pmatrix} = \vec{0}$ $\quad u_1 = 0$

$u_2 = 0$ $\quad \ker\tilde{f} = \{\vec{0}\}$ $\dim\ker\tilde{f} = 0$

$\Rightarrow$ $\dim V - \dim W = 3 - 2 = 1 = \dim\ker f - \dim\ker\tilde{f} = 1 - 0$

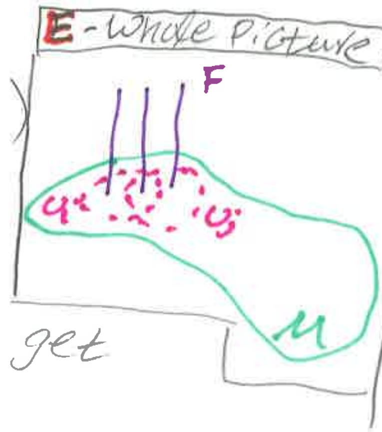
This is an example of the toy index Theorem

- What we just saw was an analytic index being equated to a topological index. This is what we want for the Dirac operator.

- Play Ground.

  - vector bundles
  - complex vector bundles
  - spin bundles

See (Nakahara)



I'll remind you of these just so you get the flavor

A differentiable fiber bundle (E, π, M, F, G)

i) A differentiable manifold (E, M, F) called (total space, base, fiber)

iv) a surjection $\pi : E \to M$ called projection

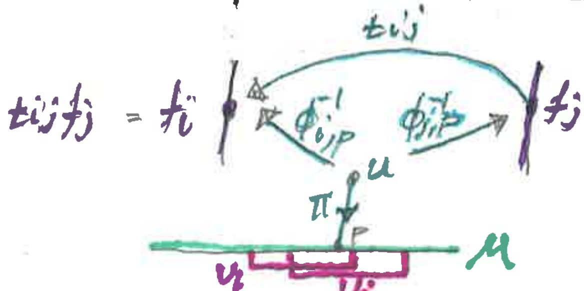
v) Lie group G called structure group
it acts on F on the left

vi) open cover $\{U_i\}$ of M with diffeo
$\phi_i : U_i \times F \to \pi^{-1}(U_i)$ s.t. $\pi \circ \phi_i(p, f) = p$
↳ local trivialization $\quad p \in M, f \in F$
since $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$
onto.

Write $\phi_i(p, f) = \phi_{ip}(f) \quad \phi_{ip}(f) : F \to F_p$, F_p fiber at p.

on $U_i \cap U_j \neq \emptyset$, $U_i, U_j \in \{U\}$
we require

$G \ni t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$



Example:
for instance let $u \in E$, $\pi(u) = p$
$\phi_{j,p}^{-1}(u) = f_j$ apply $\phi_{i,p}$ $\phi_{j,p}\phi_{j,p}^{-1} = u \Rightarrow \phi_{i,p}^{-1}\phi_{j,p}\phi_{j,p}^{-1}(u) = f_i$, $t_{ij} f_j = f_i$

Example:

Tangent Bundle -

This is an example of a vector bundle

That is $F = \mathbb{R}^k$ and $G = Gl(k, \mathbb{R})$

let M be an m-dimensional manifold

$TM = \bigcup_{p \in M} T_pM$. Where T_pM is the tangent space at p.

and $p \in U_i \subset M$. if coordinates are given by

$x^\mu = \varphi_i(p)$ a vector $V = V^\mu(p) \frac{\partial}{\partial x^\mu}\Big|_p \in T_pM$

if $\pi(u) = p$ then $\phi_i^{-1}(u) = (p, V)$ a direct product of $U_i \times \mathbb{R}^m$

if $p \in U_i \cap U_j$ then $V = V^\mu \frac{\partial}{\partial x^\mu}\Big|_p = \tilde{V}^\mu \frac{\partial}{\partial y^\mu}\Big|_p$

$$\Rightarrow \quad V^\mu \underbrace{\frac{\partial y^\nu}{\partial x^\mu}}_{Gl(m, \mathbb{R})} = \tilde{V}^\nu$$

A vector field $X \in \mathfrak{X}(M)$ may be veiwed as $X: M \to TM$ and is an example of a section and the set of all vector fields $\mathfrak{X}(M) = \Gamma(M, TM)$ - The space of sections on TM. In General $\Gamma(M, E)$ the space of sections $\sigma: M \to E$ s.t. $\pi \circ \sigma = id_M$

sections can be globally defined or only locally defined in which case it is called a local section.

Direct Sum Bundle (Whitney sum)

Suppose we have $E \xrightarrow{\pi} M$ & $E' \xrightarrow{\pi'} M'$

A product bundle $E \times E' \xrightarrow{\pi \times \pi'} M \times M'$

has typical fiber $F \oplus F'$ That is

$$f = \begin{pmatrix} v \\ w \end{pmatrix}, \quad v \in F, \ w \in F'$$

Which is clearly a vector space of dim = $\dim F + \dim F'$

under

$$\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} v+v' \\ w+w' \end{pmatrix}$$

$$\lambda \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \lambda v \\ \lambda w \end{pmatrix}$$

if $\pi(u) = p$ $\pi'(u') = p'$ Then $(u, u') \in E \times E'$

& $(\pi \times \pi')(u, u') = (p, p')$

Ex.) if $M = M_1$ & $M = M_2$ Then $TM = TM_1 \times TM_2$

i.e. The tangent bundle of M is product of The Tangent bundles of M_1 & M_2

Now let $M' = M$, $E \xrightarrow{\pi} M$ $E' \xrightarrow{\pi'} M$

$E \oplus E'$ is The Pullback bundle by The trivial map $f : M \to M \times M$ as $f(p) = (p, p)$

$$\begin{tikzcd} E \oplus E' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \times E' \arrow[d, "\pi \times \pi'"] \\ M \arrow[r, "f"] & M \times M \end{tikzcd}$$

All that This formalism says is That

$$E \oplus E' = \{ (u, u') \in E \times E' \mid (\pi \times \pi')(u, u') = (p, p) \}$$

Ex.) Normal Bundle

Consider $S^2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

We can construct the tangent space $TS^2 = \bigcup_{p \in S^2} T_pS^2$

Consider $\varphi_i = x^\mu$ as coordinates on $U_i \subset S^2$.

Then $V = V^\mu \frac{\partial}{\partial x^\mu}\Big|_p \in T_pM$; for $\pi(u) = p$

$$\phi_i^{-1}(u) = (p, V) \in S^2 \times \mathbb{R}^2, \quad V \in F_p \cong \mathbb{R}^2$$

Consider the set of vectors in $\mathbb{R}^3$ s.t.

For V embedded in $\mathbb{R}^3$ $\quad U \cdot V = 0$

The set $\{U\}$ at p consists of lines $\mathbb{R}$

that pierce the surface of S^2 and constitutes

the normal bundle $NS^2 = \bigcup_{p \in S^2} N_pS^2$, $NS^2 \cong \mathbb{R}$

Let ψ_j be the local trivializations of NS^2

Then $\psi_j^{-1}(v) = (p, U) \in S^2 \times \mathbb{R}$ provided

$\pi' : NS^2 \to S^2$ takes $\pi(v) = p$.

Construct the product bundle $TS^2 \times NS^2$

if $(u, v) \in TS^2 \times NS^2$ then $\Phi_i = \phi_i \times \psi_j$

takes $\Phi_i(u,v) = (\phi_i \times \psi_i)(u,v) = (p, p; V, U)$

Notice however I have chosen $\in S^2 \times S^2 \times \mathbb{R}^2 \times \mathbb{R}^1$

points s.t. $\pi'(v) = \pi(u) = p$ and so

we can reduce $TS^2 \times NS^2 \to TS^2 \oplus NS^2$

with

$$\Phi_i(u,v) = (p; V, U)$$

with fiber $\mathbb{R}^2 \oplus \mathbb{R} \cong \mathbb{R}^3$

with transition functions

$$T_{ij} = \begin{pmatrix} (\, t_{ij} \,)^{2\times 2} & 0 \\ & 0 \\ 0 \quad 0 & \alpha \end{pmatrix} \quad \alpha \in \mathbb{R}, \quad t_{ij} \in GL(2, \mathbb{R})$$