talk 3.

topics:

- Susy QM flat space / curved space
- Another formula for analytic index
- Path integral Expression & Elements of Proof

Susy QM:

Recall Lagrangian & Hamiltonian for traveling Eph in $\mathbb{R}^d$

$$L = \frac{1}{2}\dot{x}_k \dot{x}_k + \frac{i}{2}\psi_k \dot{\psi}_k, \quad P_k = \dot{x}_k \quad \pi_k = -i\psi_k/2$$

$$\{x_j, P_k\}_{PB} = [\psi_j, \psi_k]_{PB} = \delta_{jk}$$

$$H = \dot{x} P_j - \dot{\psi}_j \frac{i}{2}\psi_j - L = \frac{1}{2}P^2 = -\frac{1}{2}\Delta$$

The Super Symmetry transformations

$$\delta x_j = i\epsilon\psi_j \qquad \delta\psi_j = -\epsilon\dot{x}_j \qquad *$$

leave L invariant $\delta L = -\frac{i}{2}\frac{d}{dt}(\psi_j \epsilon \dot{x}_j)$

leaving us to define the noether Charge (Supercharge)

$$\epsilon Q = i\epsilon P_j \psi_j = i\epsilon\psi_j \dot{x}_j$$

Recall that Q has an interesting transformation under $*$

$$\epsilon\delta Q = i\epsilon\,\delta\psi_j \dot{x}_j + i\epsilon\psi_j \frac{d}{dt}\delta x_j$$

$$= i\epsilon(-\epsilon\dot{x}_j)\dot{x}_j + i\epsilon\psi_j \frac{d}{dt}(i\epsilon\psi_j)$$

$$= i\epsilon(-\epsilon\dot{x}_j\dot{x}_j + \psi_j i\dot{\psi}_j)$$

$$= -2i\epsilon^2\left(\frac{1}{2}\dot{x}_j\dot{x}_j + \frac{i}{2}\psi_j\dot{\psi}_j\right) = -2i\epsilon^2 L\,!$$

The variation gives back L!

Further more consider

$$\{Q,Q\} = 2Q^2 = 2\, iP_j\psi_j\,(iP_k\psi_k)$$

$$= 2\, iP_k\psi_k\,(iP_j\psi_j)$$

$$= \frac{2}{2}\, i^2 P_j P_k(\psi_j\psi_k + \psi_k\psi_j)$$

$$= -\, P_j P_k(\delta_{jk}) = -P_k P_k$$

We see that $Q^2 = -H$! $\quad = -2H$!

(x_j; L - bosonic) (ψ_j, Q - fermionic)

Lets apply some of this to a general manifold. Take M as a $\dim M = 2n$ Riemannian manifold with metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

if the x^μ are the coordinates in $U \subset M$

Then the $\psi^\mu \in T_{x(t)}M$ where i abuse notation and let t play a dual role of coordinate $x^\mu = (t, \ldots)$ and parameter $\gamma(t)$ is a path in U such that in local coordinates $\frac{d\gamma}{dt} = \dot{\vec{x}} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} = \psi^\mu \frac{\partial}{\partial x^\mu}$

Where here $\psi(t) \equiv \psi(x(t))$

Then

$$\psi^\mu \to \psi'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\psi^\nu$$

; under a susy transformation

$$\delta x'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \delta x^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} i\epsilon\psi^{\nu}$$

$$= i\epsilon\psi'^{\mu}$$

$$\delta\psi'^{\mu} = \delta\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\psi^{\nu}\right) = \delta\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\psi^{\nu} + \frac{\partial x'^{\mu}}{\partial x^{\nu}}\delta\psi^{\nu}$$

$$= \frac{\partial}{\partial x^{\nu}}(\delta x'^{\mu})\psi^{\nu} + \frac{\partial x'^{\mu}}{\partial x^{\nu}}(-\epsilon\dot{x}^{\nu})$$

$$= \frac{\partial}{\partial x^{\nu}}\left(\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\delta x^{\lambda}\right)\psi^{\nu} + -\epsilon\dot{x}'^{\mu}$$

$$= \frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x^{\lambda}}\delta x^{\lambda}\psi^{\nu} - \epsilon\dot{x}'^{\mu}$$

$$= \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x^{\lambda}} i\epsilon\psi^{\lambda}\psi^{\nu}}_{\text{even sumed over odd Thus Vanishes.}} - \epsilon\dot{x}'^{\mu} = -\epsilon\dot{x}'^{\mu}$$

We see the susy transformation is invariant under general coordinate transformations. This leads us to generalize Q to

$$Q = i\langle\dot{x},\psi\rangle = i g_{\mu\nu}\dot{x}^{\mu}\psi^{\nu}$$

Remember $\delta Q = -2i\epsilon L$

$$\delta Q = i\,\delta(g_{\mu\nu}\dot{x}^{\mu}\psi^{\nu}) = i\left[\delta g_{\mu\nu}\dot{x}^{\mu}\psi^{\nu} + g_{\mu\nu}\frac{d}{dt}\delta x^{\mu}\psi^{\nu} + g_{\mu\nu}\dot{x}^{\mu}\delta\psi^{\nu}\right]$$

$\delta g_{\mu\nu}$? $g_{\mu\nu}(x^{\mu})$ fn of coor-dinates

$\Rightarrow \delta g_{\mu\nu} = \partial_\lambda g_{\mu\nu}\delta x^{\lambda}$

we need the following trick

$$g_{\mu\alpha} g^{\mu\rho} \left[\partial_\lambda g_{\rho\nu} - \frac{1}{2} \left(\partial_\lambda g_{\nu\rho} + \partial_\rho g_{\nu\lambda} - \partial_\nu g_{\lambda\rho} \right) \right] = \Gamma^{\mu}_{\nu\lambda} g_{\mu\alpha}$$

δ^ρ_α

$$\partial_\lambda g_{\alpha\nu} - \frac{1}{2} g_{\mu\alpha} g^{\mu\rho} \left[\partial_\lambda g_{\nu\rho} + \partial_\rho g_{\nu\lambda} - \partial_\nu g_{\alpha\rho} \right] = \Gamma^{\mu}_{\nu\lambda} g_{\mu\alpha}$$

$$\boxed{\partial_\lambda g_{\alpha\nu} = 2\Gamma^{\mu}_{\nu\lambda} g_{\mu\alpha}}$$

$$\Rightarrow \delta Q = i\epsilon \left[i \partial_\lambda g_{\mu\nu} \, i\epsilon \psi^\lambda \dot{x}^\mu \psi^\nu + i g_{\mu\nu} \dot{\psi}^\mu \psi^\nu - g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]$$

$$= i\epsilon \left[i \dot{x} \cdot 2 \Gamma^{\beta}_{\nu\lambda} g_{\beta\mu} \dot{x}^\mu \psi^\lambda \psi^\nu + \ldots \right.$$

$$= -2i\epsilon \left[-i \dot{x} \Gamma^{\beta}_{\nu\lambda} g_{\beta\mu} \dot{x}^\mu \psi^\lambda \psi^\nu + \frac{1}{2} i g_{\mu\nu} \psi^\nu \dot{\psi}^\mu + \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]$$

Quick definition of some needed objects.

Connection 1-form $\Gamma^\mu{}_\nu = dx^\lambda \Gamma^\mu_{\lambda\nu}$

Curvature 2-form $\tilde{R}^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\sigma \wedge \Gamma^\sigma{}_\nu$

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in terms of $dx^\rho \wedge dx^\sigma$ $\tilde{R}^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma} dx^\rho \wedge dx^\sigma$

ordinary Riemann curvature

$$R^\kappa{}_{\lambda\mu\nu} = \langle dx^\kappa, \nabla_\mu \nabla_\nu \partial_\lambda - \nabla_\nu \nabla_\mu \partial_\lambda \rangle$$
$$= \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma_\mu{}^\kappa{}_\lambda + \Gamma^\eta_{\nu\lambda} \Gamma^\kappa_{\mu\eta} - \Gamma^\eta_{\mu\lambda} \Gamma^\kappa_{\nu\eta}$$

Finally: lets look one more time at ind D

Consider vector bundles $E_\pm \xrightarrow{\pi} M$, $E = E_+ \oplus E_-$

$D : \Gamma(M, E^+) \to \Gamma(M, E^-)$

$D^+ : \Gamma(M, E^-) \to \Gamma(M, E^+)$

$D \,\&\, D^+$ elliptic op$\triangle$s. Assuming D is Fredholm

$\text{in}D = \dim(\ker D) - \dim(\ker D^+)$

Furthermore in$D$ is invariant under small deformations of D. Proof Nakahara pg 488

Theorem 12.5 $\text{ind}D = \text{Tr}\, e^{-\beta D^+ D} - \text{Tr}\, e^{-\beta D D^+}$

$\beta > 0$ $\,\&\,$ ind$D$ is independent of β.

Proof The trace is over the eigenstates $\{\Phi_n\}$ & $\{\Psi_n\}$ & $D^\dagger D$, that is $D^\dagger D\Phi_n = \lambda_n\Phi_n$ & $\Psi_n \equiv D\Phi_n/\sqrt{\lambda_n}$

let $\{\varphi_i^0\}$ & $\{\Psi_i^0\}$ be orthonormal eigensections of $\ker D$ & $\ker D^\dagger$ respectfully with $\dim\ker D = \kappa$, $\dim\ker D^\dagger = \kappa'$

$$\mathrm{Tr}\, e^{-\beta D^\dagger D} - \mathrm{Tr}\, e^{-\beta D D^\dagger}$$

$$= \sum_{\lambda_n\neq 0}\langle\Phi_n|e^{-\beta D^\dagger D}|\Phi_n\rangle - \sum_{\lambda_n\neq 0}\langle\Psi_n|e^{-\beta DD^\dagger}|\Psi_n\rangle$$

$$+ \underbrace{\sum_{i=1}^{\kappa}\langle\varphi_i^0|\varphi_i^0\rangle}_{\kappa} - \underbrace{\sum_{i=1}^{\kappa'}\langle\Psi_i^0|\Psi_i^0\rangle}_{\kappa'}$$

$$= \sum_{\lambda_n\neq 0}\sum_{\alpha=1}^{\infty}\langle\Phi_n|(-\beta)^\alpha(D^\dagger D)^\alpha|\Phi_n\rangle - \sum_{\lambda_n\neq}\sum_{\alpha=1}^{\infty}\langle\Psi_n|(-\beta)^\alpha(DD^\dagger)|\Psi_n\rangle$$

$$+ (\kappa-\kappa')$$

$$= \sum_{\lambda_n\neq 0} e^{-\beta\lambda_n}\langle\Phi_n|\Phi_n\rangle - \sum_{\lambda_n\neq 0} e^{-\beta\lambda_n}\langle\Psi_n|\Psi_n\rangle + (\kappa-\kappa')$$

$$= \kappa-\kappa' = \dim(\ker D) - \dim(\ker D^\dagger)$$
$$= \mathrm{ind}\, D$$

additionally we see $\mathrm{ind}\, D$ is independent of β (Tr is the heat kernel)

Suppose $E = E_+ \oplus E_-$ with

$$iQ = \begin{pmatrix} 0 & D^\dagger \\ D & 0\end{pmatrix} : E \to E$$

Then the Hamiltonian $H \equiv (iQ)^2 = \begin{pmatrix} D^\dagger D & 0 \\ 0 & DD^\dagger\end{pmatrix}$

define also $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$

Then $\mathrm{ind}\, D = \mathrm{Tr}(\Gamma e^{-\beta H}) = \mathrm{Tr}\left[\begin{pmatrix} e^{-\beta D^\dagger D} & 0 \\ 0 & e^{-\beta DD^\dagger}\end{pmatrix}\right] = \mathrm{Tr}\left[e^{-\beta D^\dagger D}\right] - \mathrm{Tr}\left[e^{-\beta DD^\dagger}\right]$

Our case: Spin Bundle

M - spin manifold that is $w_2(M)$ second Stifel-Whitney class is trivial. This means $SO(k) \to$ lifted to SPIN bundle as

$$\begin{array}{ccc} SO(k) & \longrightarrow & SPIN(k) \\ \downarrow \pi & & \\ M & & \end{array}$$

$E = \Delta(M)$ - is this spin bundle

Associated with $\Delta(M)$ is the Clifford Algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$

Chirality op: $\gamma_{2n+1} \equiv i^n \gamma_1 \gamma_2 \cdots \gamma_{2n}$

$(\gamma_{2n+1})^2 = I \Rightarrow \pm 1$ eigenvalues

Decompose: $\Gamma(M, \Delta) = \Gamma(M, \Delta^+) \oplus \Gamma(M, \Delta^-)$

$$\gamma_{2n+1} \psi^\pm = \pm \psi^\pm$$

Assign: **Fermion #** $F = 0$ for $\psi \in \Gamma(M, \Delta^+)$

$F = 1$ for $\psi \in \Gamma(M, \Delta^-)$

The Γ matrix on previous page is then $\Gamma = (-1)^F$

$\{Q, \Gamma\}\psi = Q\Gamma\psi + \Gamma Q \psi$, Put $\psi = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}$

$Q\psi + \Gamma \begin{pmatrix} 0 \\ D^+\psi^+ \end{pmatrix}$

$\begin{pmatrix} 0 \\ D^+\psi^+ \end{pmatrix} + \begin{pmatrix} 0 \\ -D^+\psi^+ \end{pmatrix} = 0$

Same for $\psi \in \begin{pmatrix} 0 \\ \psi^- \end{pmatrix}$

$\{Q, \Gamma\} = 0$ since $\Gamma = \gamma_{2n+1}$

$\Rightarrow \{Q, \gamma_{2n+1}\} = 0$ leads us to write

$D = D \quad D^+ = D^+$

SUSY & The Dirac Index:

let us consider our previously defined Dirac operator Q on M with $\dim M = 2n$

With $t \to -it$ (wick rotation)

Recall $H = (iQ)^2 = \frac{1}{2} g_{\mu\nu} P^\mu P^\nu$

Then $$\operatorname{ind} Q = \operatorname{Tr} \Gamma e^{-\beta H} = \operatorname{Tr} (-1)^F e^{-\beta H}$$
$$= \int_{PBC} Dx\, D\psi\, e^{-\int_0^\beta dt\, L}$$

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \psi^\mu \frac{D\psi^\nu}{Dt}$$

PBC: Periodic Boundary Conditions

Why? Consider

$$\operatorname{tr} (-1)^F e^{-\beta H} = \sum_n \langle n | (-1)^F e^{-\beta H} | n \rangle$$

Recall $\mathcal{H} = \operatorname{span}\{|0\rangle, |1\rangle\}$ for fermionic oscillator.

Then $|f\rangle = |0\rangle f_0 + |1\rangle f_1$, $f_0, f_1 \in \mathbb{C}$

$$|\theta\rangle = |0\rangle + |1\rangle \theta$$
$$\langle\theta| = \langle 0| + \theta^* \langle 1| \quad \Big\} \ \theta, \theta^* \text{ Grassman \#'s}$$
$$c|\theta\rangle = |0\rangle\theta = |\theta\rangle\theta$$
$$\langle\theta| c^\dagger = \theta^* \langle 0| = \theta^* \langle\theta|$$
$$\langle\theta'|\theta\rangle = (\langle 0| + \theta'^* \langle 1|)(|0\rangle + |1\rangle\theta)$$
$$= \langle 0|0\rangle + \theta'^* \langle 1|1\rangle \theta = 1 + \theta'^*\theta = e^{\theta'^*\theta}$$
$$\langle\theta|f\rangle = (\langle 0| + \theta^*\langle 1|)(|0\rangle f_0 + |1\rangle f_1)$$
$$= \langle 0|0\rangle f_0 + \theta^* \langle 1|1\rangle f_1 = f_0 + \theta^* f_1$$
$$\langle\theta| c^\dagger |f\rangle = \langle\theta|1\rangle f_0 = \theta^* f_0$$
$$\langle\theta| c |f\rangle = \langle\theta|0\rangle f_1 = f_1$$

Completeness

$$\int d\theta^* d\theta\, |\theta\rangle\langle\theta| e^{-\theta^*\theta} = I$$

$$\int d\theta^* d\theta\, (|0\rangle + |1\rangle\theta)(\langle 0| + \theta^*\langle 1|)(1-\theta^*\theta)$$

$$\int d\theta^* d\theta\, (|0\rangle\langle 0| + |1\rangle\theta\langle 0| + |0\rangle\theta^*\langle 1| + |1\rangle\theta\theta^*\langle 1|)(1-\theta^*\theta)$$

$$(|0\rangle\langle 0| + |1\rangle\theta\langle 0| + |0\rangle\theta^*\langle 1| + |1\rangle\theta\theta^*\langle 1|)$$

$$-(|0\rangle\langle 0|\theta^*\theta + |1\rangle\theta\langle 0|\theta^*\theta + |0\rangle\theta^*\langle 1|\theta^*\theta)$$

$$-(-|0\rangle\theta\theta^*\langle 0|)$$

only non zero $\int d\theta^* d\theta\, (|1\rangle\theta\theta^*\langle 1| + |0\rangle\theta\theta^*\langle 0|)$

$$= |1\rangle\langle 1| + |0\rangle\langle 0| = I$$

$$\Rightarrow Z = \sum_n \langle n| e^{-\beta H} |n\rangle$$

$$= \sum_n \int d\theta^* d\theta\, e^{-\theta^*\theta} \langle n|\theta\rangle\langle\theta| e^{-\beta H}|n\rangle$$

$$= \sum_n \int d\theta^* d\theta (1-\theta^*\theta)[\langle n|0\rangle + \langle n|1\rangle\theta][\langle 0|e^{-\beta H}|n\rangle + \theta^*\langle 1|e^{-\beta H}|n\rangle]$$

$$= \sum_n \int d\theta^* d\theta (1-\theta^*\theta)[\langle n|0\rangle\langle 0|e^{-\beta H}|n\rangle + \langle n|0\rangle\theta^*\langle 1|e^{-\beta H}|n\rangle$$

$$+ \langle n|1\rangle\theta\langle 0|e^{-\beta H}|n\rangle + \langle n|1\rangle\theta\theta^*\langle 1|e^{-\beta H}|n\rangle]$$

Rearrange

$$[\langle 0|e^{-\beta H}|n\rangle\langle n|0\rangle - \theta^*\theta\langle 1|e^{-\beta H}|n\rangle\langle n|1\rangle$$

$$+ \theta\langle 0|e^{-\beta H}|n\rangle\langle n|1\rangle + \theta^*\langle 1|e^{-\beta H}|n\rangle\langle n|0\rangle]$$

Notice $|-\theta\rangle = |0\rangle - |1\rangle\theta$ will give zero.

flip its sign.

$$= \sum_n \int d\theta^* d\theta\, e^{-\theta^*\theta} \Big[\langle 0| e^{-\beta H}|n\rangle\langle n|0\rangle - \theta^*\theta \langle 1|e^{-\beta H}|n\rangle\langle n|1\rangle$$
$$+ \theta \langle 0| e^{-\beta H}|n\rangle\langle n|1\rangle - \theta^* \langle 1| e^{-\beta H}|n\rangle\langle n|0\rangle \Big]$$

$$= \int d\theta^* d\theta\, e^{-\theta^*\theta} \Big[\langle 0|e^{-\beta H}|0\rangle - \theta^*\theta\langle 1|e^{-\beta H}|1\rangle$$
$$+ \theta\langle 0|e^{-\beta H}|1\rangle - \theta^*\langle 1|e^{-\beta H}|0\rangle \Big]$$

$$= \int d\theta^* d\theta\, e^{-\theta^*\theta} \Big[\big(\langle 0| - \theta^*\langle 1| \big) e^{-\beta H} \big(|0\rangle + |1\rangle\theta \big) \Big]$$

$$= \int d\theta^* d\theta\, e^{-\theta^*\theta} \langle -\theta| e^{-\beta H}|\theta\rangle = Z$$ (partition fn for fermionic oscillator)

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back to it

$$\mathrm{tr}(-1)^F e^{-\beta H} = \sum_n \langle n|(-1)^F e^{-\beta H}|n\rangle$$ - using the above

$$= \int d\theta^* d\theta \langle -\theta|(-1)^F e^{-\beta H}|\theta\rangle e^{-\theta^*\theta}$$

noting that $(-1)^F|\theta\rangle = |0\rangle - |1\rangle\theta = |-\theta\rangle$

we get

$$= \int d\theta^* d\theta \langle \theta| e^{-\beta H}|\theta\rangle e^{-\theta^*\theta}$$

$\therefore$ we have periodic boundary conditions not anti periodic conditions

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Continuing on the SUSY trans. Wick rotate

$$\delta x^\mu = i\epsilon\psi^\mu \qquad \delta\psi = -i\epsilon\dot{x}^\mu$$

our action

$$\int_0^\beta dt \left[\frac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2} g_{\mu\nu}\psi^\mu \frac{D\psi^\nu}{Dt} \right]$$

can be manipulated a bit since we can consider $\beta \to 0$

Since ind Q is independent of β

$\to$ $t = \beta s$ $\quad dt = \beta ds \quad \frac{d}{dt} = \frac{1}{\beta}\frac{d}{ds}$

$$\int_0^1 ds\,\beta\left[\frac{1}{2} g_{\mu\nu}\frac{1}{\beta^2}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2} g_{\mu\nu}\psi^\mu\left[\frac{1}{\beta}\frac{d\psi^\nu}{dt} + \Gamma^\nu_{\lambda\alpha}\dot{x}^\lambda\psi^\alpha\frac{1}{\beta}\right]\right]$$

$$\int_0^1 ds\left[\frac{1}{2}\frac{1}{\beta} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2} g_{\mu\nu}\psi^\mu\frac{D\psi^\nu}{Dt}\right]$$

$\therefore$ x^μ = Constant, is main contribution

$x^\mu \neq$ constant exponentially small contribution.

first Extremum of action (be careful with sign flips due to Grassmann ψ)

$$\frac{\partial L}{\partial \psi^\rho} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}^\rho}\right) = 0 \quad \boxed{= \frac{D\psi^\mu}{Dt}} \text{ answer.}$$

Proof $\to$

$$= \frac{1}{2} g_{\mu\nu}\delta^\mu_\rho\frac{D\psi^\nu}{Dt} + \frac{d}{dt}\left(\frac{1}{2} g_{\mu\nu}\psi^\mu(\delta^\nu_\rho)\right) - \frac{1}{2} g_{\mu\nu}\psi^\mu\Gamma^\nu_{\lambda\alpha}\dot{x}^\lambda\delta^\alpha_\rho$$

$$= \frac{1}{2} g_{\rho\nu}\frac{D\psi^\nu}{Dt} - \frac{1}{2} g_{\alpha\nu}\psi^\alpha\Gamma^\nu_{\lambda\rho}\dot{x}^\lambda + \frac{1}{2}\partial_\lambda g_{\nu\rho}\dot{x}^\lambda\psi^\nu + \frac{1}{2} g_{\nu\rho}\frac{d\psi^\nu}{dt}$$

mult by $g^{\rho\mu}$

$$= \frac{1}{2}\left[\delta^\mu_\nu\frac{D\psi^\nu}{Dt} - g^{\rho\mu}g_{\alpha\nu}\Gamma^\nu_{\lambda\rho}\dot{x}^\lambda\psi^\alpha + g^{\mu\rho}\partial_\lambda g_{\nu\rho}\dot{x}^\lambda\psi^\nu + \delta^\mu_\nu\frac{d\psi^\nu}{dt}\right.$$

$$= \frac{1}{2}\left[\frac{D\psi^\mu}{Dt} - \underbrace{g^{\rho\mu}g_{\alpha\nu}\Gamma^\nu_{\lambda\rho}\dot{x}^\lambda\psi^\alpha + g^{\mu\rho}\partial_\lambda g_{\nu\rho}\dot{x}^\lambda\psi^\nu}_{} + \frac{d\psi^\mu}{dt}\right]$$

use an identity

$$g^{\mu\rho}\left[\partial_\lambda g_{\nu\rho} - g_{\nu\beta}\Gamma^\beta_{\lambda\rho}\right]\dot{x}^\lambda\psi^\nu$$

$$\to g^{\mu\rho}\left[\partial_\lambda g_{\nu\rho} - g_{\nu\beta}\frac{1}{2}g^{\beta\delta}(\partial_\rho g_{\delta\lambda} + \partial_\lambda g_{\rho\delta} - \partial_\delta g_{\lambda\rho})\right]$$

$$= g^{\mu\rho}\left[\frac{1}{2}(\partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\rho\nu} - \partial_\nu g_{\lambda\rho})\right] = \Gamma^\mu_{\nu\lambda}$$

$$= \frac{1}{2}\left[\frac{D\psi^\mu}{Dt} + \Gamma^\mu_{\nu\lambda}\dot{x}^\lambda\psi^\nu + \frac{d\psi^\mu}{dt}\right] = \frac{D\psi^\mu}{Dt} = 0 \checkmark$$

✓ Check This

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} = 0$$

$$\frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + \frac{1}{2}\partial_\mu g_{\alpha\beta}\psi^\alpha\frac{D\psi^\beta}{Dt} + \frac{1}{2}g_{\alpha\beta}\psi^\alpha\partial_\mu\Gamma^\beta_{\lambda\gamma}\dot{x}^\lambda\psi^\gamma$$

$$-\frac{d}{dt}\left(\frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha 2\delta^\beta_\mu + \frac{1}{2}g_{\alpha\beta}\psi^\alpha\Gamma^\beta_{\lambda\gamma}\delta^\lambda_\mu\psi^\gamma\right)$$

$$= \frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + \frac{1}{2}\partial_\mu g_{\alpha\beta}\psi^\alpha(\dot{\psi}^\beta + \Gamma^\beta_{\lambda\nu}\dot{x}^\lambda\psi^\nu)$$

$$+ \frac{1}{2}g_{\alpha\beta}\psi^\alpha\partial_\mu\Gamma^\beta_{\lambda\gamma}\dot{x}^\lambda\psi^\gamma - \frac{1}{2}\Big[2\partial_\lambda g_{\alpha\mu}\dot{x}^\alpha\dot{x}^\lambda + 2g_{\alpha\mu}\ddot{x}^\alpha$$

$$+ \partial_\lambda g_{\alpha\beta}\dot{x}^\lambda\psi^\alpha\Gamma^\beta_{\mu\gamma}\psi^\gamma + g_{\alpha\beta}\dot{\psi}^\alpha\Gamma^\beta_{\mu\gamma}\psi^\gamma$$

$$+ g_{\alpha\beta}\psi^\alpha\partial_\lambda\Gamma^\beta_{\mu\gamma}\dot{x}^\lambda\psi^\gamma + g_{\alpha\beta}\psi^\alpha\Gamma^\beta_{\mu\gamma}\dot{\psi}^\gamma\Big]$$

$$= -g_{\alpha\mu}\ddot{x}^\alpha + \frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - \partial_\lambda g_{\alpha\mu}\dot{x}^\alpha\dot{x}^\lambda$$

$$+ \frac{1}{2}\psi^\alpha\dot{x}^\lambda\psi^\nu\Big[\partial_\mu g_{\alpha\beta}\Gamma^\beta_{\lambda\nu} + g_{\alpha\beta}\partial_\mu\Gamma^\beta_{\lambda\nu} - g_{\alpha\beta}\partial_\lambda\Gamma^\beta_{\mu\nu}$$

$$- \partial_\lambda g_{\alpha\beta}\Gamma^\beta_{\mu\nu}\Big] + \frac{1}{2}\partial_\mu g_{\alpha\beta}\psi^\alpha\dot{\psi}^\beta - \frac{1}{2}g_{\alpha\beta}\dot{\psi}^\alpha\Gamma^\beta_{\mu\gamma}\psi^\gamma$$

$$+ \frac{1}{2}g_{\alpha\beta}\psi^\alpha\Gamma^\beta_{\mu\gamma}\dot{\psi}^\gamma$$

There are alot of terms here.
First Do The Square bracket
and use

$$\partial_\lambda g_{\alpha\nu} = 2g_{\mu\alpha}\Gamma^\mu_{\lambda\nu}$$

$$\frac{1}{2}\psi^\alpha\dot{x}^\lambda\psi^\nu\left[2g_{\gamma\alpha}\Gamma^\gamma_{\mu\beta}\Gamma^\beta_{\lambda\nu} + g_{\alpha\beta}\partial_\mu\Gamma^\beta_{\lambda\nu} - g_{\alpha\beta}\partial_\lambda\Gamma^\beta_{\mu\nu} - 2g_{\gamma\alpha}\Gamma^\gamma_{\lambda\beta}\Gamma^\beta_{\mu\nu}\right]$$

$$\frac{1}{2}\psi^\alpha\dot{x}^\lambda\psi^\nu\left[R_{\alpha\nu\mu\lambda} + g_{\alpha\beta}(\Gamma^\beta_{\mu\delta}\Gamma^\delta_{\lambda\nu} - \Gamma^\beta_{\lambda\delta}\Gamma^\delta_{\mu\nu})\right]$$

now the second & third term on first line

$$\frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - \partial_\beta g_{\alpha\mu}\dot{x}^\alpha\dot{x}^\beta$$

$$g_{\alpha\lambda}\Gamma^\lambda_{\mu\beta}\dot{x}^\alpha\dot{x}^\beta - 2 g_{\alpha\lambda}\Gamma^\lambda_{\beta\mu}\dot{x}^\alpha\dot{x}^\beta$$

$$= - g_{\alpha\lambda}\Gamma^\lambda_{\beta\mu}\dot{x}^\alpha\dot{x}^\beta$$

This Page is incomplete i am not going to finish it. Whats missing is to show

now final 3 terms

$$\frac{1}{2}\partial_\mu g_{\alpha\beta}\psi^\alpha\dot{\psi}^\beta - \frac{1}{2}g_{\alpha\beta}\Gamma^\beta_{\mu\gamma}(\dot{\psi}^\alpha\psi^\gamma + \psi^\alpha\dot{\psi}^\gamma) = 0$$

$$g_{\alpha\beta}\Gamma^\beta_{\mu\gamma}\psi^\alpha\dot{\psi}^\gamma - \frac{1}{2}g_{\alpha\beta}\Gamma^\beta_{\mu\gamma}(\dot{\psi}^\alpha\psi^\gamma + \psi^\alpha\dot{\psi}^\gamma)$$

$$-\frac{1}{2}g_{\alpha\beta}\Gamma^\beta_{\mu\gamma}(\dot{\psi}^\alpha\psi^\gamma - \psi^\alpha\dot{\psi}^\gamma)$$

$$\frac{1}{2}\psi^\gamma\dot{x}^\lambda\psi^\nu g_{\gamma\beta}(\Gamma^\beta_{\mu\delta}\Gamma^\delta_{\lambda\nu} - \Gamma^\beta_{\lambda\delta}\Gamma^\delta_{\mu\nu})$$

$$\Gamma^\beta_{\mu\delta}\Gamma^\delta_{\lambda\nu}$$

$$-\dot{x}^\lambda\Gamma^\alpha_{\lambda\nu}\psi^\nu\psi^\gamma + \psi^\alpha\dot{x}^\lambda\Gamma^\gamma_{\lambda\nu}\psi^\nu$$

$$\dot{x}^\lambda(+\Gamma^\alpha_{\lambda\nu}\psi^\gamma + \psi^\alpha\Gamma^\gamma_{\lambda\nu}\psi^\nu$$

$$\frac{1}{2}\psi^\gamma\dot{x}^\lambda\psi^\nu(g_{\gamma\beta}\Gamma^\beta_{\mu\delta}\Gamma^\delta_{\lambda\nu} + g_{\alpha\beta}\Gamma^\beta_{\mu\gamma}\Gamma^\alpha_{\lambda\nu}$$

$$\psi^\nu\psi^\gamma + \ldots$$

$$\dot{x}^\lambda(g_{\alpha\beta}\Gamma^\beta_{\mu\gamma}\Gamma^\alpha_{\lambda\nu} + g_{\alpha\beta}\Gamma^\beta_{\mu\delta}\Gamma^\delta_{\lambda\nu})\psi^\gamma\psi^\nu$$

$$\dot{x}^\lambda\Gamma^\alpha_{\lambda\nu}\psi^\nu \qquad g_{\alpha\beta}\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\lambda\nu}$$

$$-\psi^\gamma\dot{\psi}^\alpha g_{\gamma\beta}\Gamma^\beta_{\mu\alpha} + \psi^\gamma g_{\gamma\beta}\Gamma^\beta_{\lambda\delta} \qquad \dot{x}^\lambda\partial_\lambda g^{\rho\sigma}\Gamma^\mu_{\rho\nu}\psi^\nu$$

check indices

to evaluate the path integral we need the flucuation ops.

around $\psi_0^\mu = \psi^\mu$ $x_0^\mu = x^\mu$ the classical solutions

So put $\psi^\mu = \psi_0^\mu + \eta^\mu$; $x^\mu = x_0^\mu + \xi^\mu$ (1)

To make life easier use Riemann normal Coordinates that $g_{\mu\nu}|_{x=x_0} = \delta_{\mu\nu}$ $\partial_\lambda g_{\mu\nu}|_{x=x_0} = 0$ (2)

in particular this means $\Gamma^\lambda_{\mu\nu}|_{x=x_0} = 0$

This does not mean $R^\mu{}_{\sigma\lambda\alpha} = 0$

to calulate the second order flucuation ops we need to act $\frac{\delta}{\delta x^\mu}$; $\frac{\delta}{\delta \psi^\mu}$ on the respective Euler eqn. Then the equations are given by (symbolically)

$\xi D\xi + \eta D\eta$ with D the fluc.

x: $-g_{\lambda\mu}\left(\frac{d\dot{x}^\mu}{dt} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right) + \frac{1}{2}R_{\mu\nu\lambda\rho}\psi^\mu\psi^\nu\dot{x}^\rho$

First evaluate with (1) & (2)

$$-\delta_{\lambda\mu}\frac{d^2}{dt^2}(x_0^\mu + \xi^\mu) + \frac{1}{2}R_{\mu\nu\lambda\rho}(\psi_0^\mu + \eta^\mu)(\psi_0^\nu + \eta^\nu)\frac{d}{dt}(x_0^\rho + \xi^\rho)$$

$$-\delta_{\lambda\mu}\frac{d^2\xi^\mu}{dt^2} + \frac{1}{2}R_{\mu\nu\lambda\rho}\psi_0^\mu\psi_0^\nu\dot{\xi}^\rho + \frac{1}{2}R_{\mu\nu\lambda\rho}\psi_0^\mu\eta^\nu\dot{\xi}^\rho$$

2nd order

$$+ \underbrace{\frac{1}{2}R_{\mu\nu\lambda\rho}\eta^\mu\psi_0^\nu\dot{\xi}^\rho}_{\text{3rd order}} + \underbrace{\frac{1}{2}R_{\mu\nu\lambda\rho}\eta^\mu\eta^\nu\dot{\xi}^\rho}_{\text{4th order}}$$

$\boxed{\text{orders when thinking of } \xi D\xi}$

fluc op is then

$$-\delta_{\lambda\mu}\frac{d^2}{dt^2} + \frac{1}{2}R_{\mu\nu\lambda\rho}\psi_0^\mu\psi_0^\nu = -\delta_{\lambda\nu}\frac{d^2}{dt^2} + \tilde{R}_{\lambda\rho}\delta_{\rho\nu}\frac{d}{dt}$$

eqn is $\langle \xi, D\xi \rangle$

$$= -\xi^\mu\frac{d^2\xi^\mu}{dt^2} + R_{\lambda\nu}\xi^\lambda\dot{\xi}^\nu$$

for ψ
$$\frac{D\psi^\mu}{Dt} = 0$$

$$\frac{D(\psi_0^\mu + \eta^\mu)}{Dt} = 0$$

$$\frac{D\eta^\mu}{Dt} = \frac{d\eta^\mu}{dt} + \underset{0}{\cancel{\Gamma^\mu_{\alpha\beta}\psi^\alpha \mathring{x}^\beta}} = 0$$

$$= \frac{d\eta^\mu}{dt}, \text{ fluc op is } \frac{d}{dt}$$

eqm is
$$\langle \eta, D\eta \rangle = \eta^\mu \frac{d\eta^\mu}{dt}$$

The second order action is then
$$S_2 = \int_0^1 dt \; \xi^\mu \frac{d^2}{dt^2}\xi^\mu - \xi^\mu \tilde{R}_{\mu\nu}\xi^\nu + \eta^\mu \frac{d}{dt}\eta^\mu$$

With ops.
$$D_\xi = \delta_{\nu\mu}\frac{d^2}{dt^2} - \tilde{R}_{\nu\mu}\frac{d}{dt}$$
$$D_\eta = \frac{d}{dt}$$

Currently we have

$$\mathrm{ind}\,Q = N \int D\eta\, D\xi\, e^{-S_2}$$

lets decompose η ; ξ as

$$\eta^\mu = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \eta^\mu_n e^{i2\pi n t/\beta}$$

$$\xi^\mu = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \xi^\mu_n e^{i2\pi n t/\beta}$$

and since we have the flucuation operators we can write down the "solution"

$$D\eta = \prod_{n=-\infty}^{\infty} \prod_{\mu=1}^{d} d\eta^\mu_n \,, \quad D\xi = \prod_{n=-\infty}^{\infty} \prod_{\mu=1}^{d} d\xi^\mu_n$$

We will seperately calculate the zero modes $d\eta^\mu_0$ $d\xi^\mu_0$

$$\mathrm{ind}\,Q = N \int \prod_{n=-\infty}^{\infty} \prod_{\mu=1}^{d} d\eta^\mu_n\, d\xi^\mu_n\, e^{-\int dt\, \xi^\mu \left(\delta_{\mu\nu}\frac{d^2}{dt^2} - \tilde{R}_{\mu\nu}\frac{d}{dt}\right)\xi^\nu + \eta^\mu \frac{d}{dt}\eta^\mu}$$

and recall that the gaussian integrals are solved via a functional determinent

$$\mathrm{ind}\,Q = N \int \prod_{\nu=1}^{d} d\eta^\mu_0\, d\xi^\mu_0 \left[\det\left(\delta_{\mu\nu}\frac{d^2}{dt^2} - \tilde{R}_{\mu\nu}\frac{d}{dt}\right)\right]^{-1/2} \left[\det\left(\delta_{\mu\nu}\frac{d}{dt}\right)\right]^{1/2}$$

$$N\int \prod_{n=-\infty}^{\infty} \prod_{\mu=1}^{d} d\eta_n^\mu \, d\xi_n^\mu \, \underbrace{e^{-\int dt \left(\xi^\mu \left(g_{\mu\nu}\frac{d^2}{dt^2} - \tilde{R}_{\mu\nu}\frac{d}{dt}\right)\xi^\nu + \eta^\mu \frac{d}{dt}\eta^\mu\right)}}$$

$$e^{-\int_0^\beta \frac{dt}{\beta} \sum_{m,n=-\infty}^{\infty} \xi_n^\mu \xi_m^\mu \left(e^{\frac{i2\pi nt}{\beta}}\left(\frac{i2\pi m}{\beta}\right)^2 e^{\frac{i2\pi mt}{\beta}}\right) - \tilde{R}_{\mu\nu}\xi_n^\mu \xi_m^\nu \left(\frac{i2\pi m}{\beta}\right) e^{\frac{i2\pi t(n+m)}{\beta}}}$$

$$+ \eta_n^\mu \eta_m^\mu \left(\frac{i2\pi m}{\beta}\right) e^{\frac{i2\pi t(n+m)}{\beta}}$$

$$+ \xi_n^\mu \xi_{-n}^\mu \left(-\left(\frac{i2\pi n}{\beta}\right)^2\right) - \tilde{R}_{\mu\nu}\xi_n^\mu \xi_{-n}^\nu \left(\frac{i2\pi n}{\beta}\right)$$

$$- i2\pi n \, \eta_n^\mu \eta_{-n}^\mu$$

$$e^{\xi_n^\mu \xi_{-n}^\mu \left(\frac{4\pi^2 n^2}{\beta^2}\right) - \tilde{R}_{\mu\nu}\xi_n^\mu \xi_{-n}^\nu \left(\frac{i2\pi n}{\beta}\right) - \frac{i2\pi n}{\beta}\eta_n^\mu \eta_{-n}^\mu}$$

$$\cdots e^{\xi_1^\mu \xi_{-1}^\mu \left(\frac{4\pi^2 1^2}{\beta^2}\right) - \tilde{R}_{\mu\nu}\xi_1^\mu \xi_{-1}^\nu \left(\frac{i2\pi}{\beta}\right) - \frac{i2\pi}{\beta}\eta_1^\mu \eta_{-1}^\mu}$$

$\xi_1^0 \xi_{-1}^0 + \xi_1^1 \xi_{-1}^1 \cdots \cdots \times \cdots\cdots$

or put in $\sum_{m=-\infty}^{\infty}$ $m \to -m$ $\sum_{m=-\infty}^{\infty}$

and $\xi_1^\mu \xi_1^\mu + \cdots\cdots\cdots$

This is nothing more then a guassian integration and we have a product of the eigenvalues

$$\Rightarrow N\int \prod_{\mu=1}^{d} d\eta_0^\mu \frac{d\xi_0^\mu}{\sqrt{2\pi}} \left[\mathrm{Det}\left(-g_{\mu\nu}\frac{d}{dt} + \tilde{R}_{\mu\nu}(x_0)\right)\right]^{-1/2}$$

doesn't include zero modes.

to continue we need

$N = N_b N_f$ The normalization

This can be calculated from the free expressions

$$= \int D\psi \, e^{-\frac{1}{2}\int_0^1 \psi \cdot \dot{\psi} \, dt}$$

$$= N_f \, \mathrm{Det}(\delta_{\mu\nu} \partial_t)^{1/2} \int d\psi_0^1 \cdots d\psi_0^{2n}$$

$\vdots$

i am skipping this
i have run out of time
for these seminar talks
see nakahara

$\rightarrow \quad N_f = i^n$

like wise

$$\int Dx^\mu \, e^{-\frac{1}{2}\int_0^1 \dot{x}^\mu \dot{x}^\mu} = N_b \frac{1}{(\mathrm{Det}(-\delta_{\mu\nu}\partial_t^2))} \int \frac{dx_0^1}{\sqrt{2\pi}} \cdots dx_0^{2n}$$

$\vdots$

$N_b = 1$

$$\rightarrow \ \mathrm{ind}\, Q = i^n \int \prod_{\mu=1}^{d} \frac{d\xi_0^\mu}{\sqrt{2\pi}} \, d\eta_0^\mu \left(\mathrm{Det}\left(-\delta_{\mu\nu} \frac{d}{dt} + \tilde{R}_{\mu\nu}(t_0) \right) \right)^{-1/2}$$

now we need

notice

$$\tilde{R}_{\mu\nu} = \frac{1}{2\pi} \tilde{R}_{\mu\nu\rho\sigma} \psi_0^\rho \psi_0^\sigma$$
$$= \frac{-1}{2\pi} \tilde{R}_{\mu\nu\rho\sigma} \psi_0^\sigma \psi_0^\rho$$
$$= -\frac{1}{2\pi} \tilde{R}_{\mu\nu\sigma\rho} \psi_0^\rho \psi_0^\sigma$$
$$= \frac{1}{2\pi} \tilde{R}_{\mu\nu\rho\sigma} \psi_0^\rho \psi_0^\sigma = \tilde{R}_{\mu\nu}$$

i.e. its grassmann even

and $\tilde{R}_{\mu\nu} = -\tilde{R}_{\nu\mu}$ This means we can

Write (in even dim manifolds)

$$\tilde{R}_{\mu\nu} = \begin{pmatrix} 0 & y_1 & & & \\ -y_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & y_n \\ & & & -y_n & 0 \end{pmatrix}$$

Then $-\delta_{\mu\nu}\frac{d}{dt} + \tilde{R}_{\mu\nu}$ in the first block looks like

$$\det\begin{pmatrix} -d/dt & y_1 \\ -y_1 & -d/dt \end{pmatrix} = \det\left(\frac{d^2}{dt^2} + y_1^2\right)$$

$$= \prod_{n\neq 0}\left(y_1^2 - \left(\frac{2\pi n}{\beta}\right)^2\right)$$

$$= \left[\prod_{n\geq 1}\left(\frac{2\pi n}{\beta}\right)^2 \prod_{n\geq 1}\left(1 - \frac{y_1^2\beta^2}{(2\pi n)^2}\right)\right]^2$$

$$= \left(\frac{\sin \beta y_1/2}{y_1/2}\right)^2$$

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\cdots = \prod_{n\geq 1}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

but we have n blocks so

$$\mathrm{Det}(\quad) = \prod_{j=1}^{n} \frac{y_j/2}{\sin \beta y_j/2}$$

This is written formally as

$$\frac{1}{\beta^{d/2}} \det\left(\frac{\beta\tilde{R}}{\sin\beta\tilde{R}/2}\right)^{1/2}$$

$$\mathrm{ind}\, Q = i^n \int \prod_{\mu=1}^{2n} \frac{d\xi_0^\mu}{\sqrt{2\pi}} \frac{d\eta_0^\mu}{\beta^{d/2}} \det\left(\frac{\beta\tilde{R}}{\sin\beta\tilde{R}/2}\right)^{1/2}$$

Some formal tricks let us change variables and erase the apparent β dependence

Atiyah Singer index Theorem for Dirac op.

$$\mathrm{ind}\, Q = i^n \int \prod_{\mu=1}^{2n} dx_0^\mu\, d\chi_0^\mu \det\left(\frac{\frac{1}{8\pi} R_{\mu\nu\rho\sigma}\chi_0^\rho\chi_0^\sigma}{\sin\left(\frac{1}{8\pi}R_{\mu\nu\rho\sigma}\chi_0^\rho\chi_0^\sigma\right)}\right)^{1/2} = \nu_0^+ - \nu_0^-$$