HEP seminar talk # 2

topics:

- Fredholm operators
- Symbols
- Dirac operator & the spin complex

Fredholm operators

Consider a vector bundle $E \xrightarrow{\pi} M$ & $F \xrightarrow{\pi'} M$

Denote by $\Gamma(M,E)$ the space of sections of E. That is $\Gamma(M,E) \equiv \{\sigma \mid \sigma : M \to E \;\&\; \pi \circ \sigma = id_M\}$
identity

Then an operator is map on the space of sections $D : \Gamma(M,E) \longrightarrow \Gamma(M,F)$

if we define a fiber metric E & F then we can define the adjoint $D^{+} : \Gamma(M,F) \to \Gamma(M,E)$

- D is a differential op. so it has analytic information
  - spectrum
  - degeneracy

As before zero eigenvectors or eigensections constitute the kernel

$$\ker D \equiv \{ s \in \Gamma(M,E) \mid Ds = 0 \}$$

$$\ker D^{+} \equiv \{ s \in \Gamma(M,F) \mid D^{+}s = 0 \}$$

The analytic index is

$$\mathrm{ind}(D) \equiv \dim(\ker D) - \dim(\ker D^{+})$$

first we need Elliptic operators

as before let $D : \Gamma(M, E) \to \Gamma(M, F)$

take a chart $U \subset M$; coordinates on that chart x^μ

if $\dim E = k$ $\dim F = k'$ The most general form of D is

$$[D s(x)]^\alpha = \sum_{|M| \leq N} A^{M\alpha}{}_a D_M s^a , \quad \begin{array}{l} a = 1, \dots k \\ \alpha = 1, \dots k' \\ M = (\mu_1, \mu_2, \dots \mu_j) \\ \mu_j \in \mathbb{Z}_+ \\ |M| = \mu_1 + \mu_2 + \dots \mu_j \end{array}$$

With $D = \dfrac{\partial^{|M|}}{(\partial x^1)^{\mu_1} \cdots (\partial x^m)^{\mu_m}}$

This looks confusing. And it is at first. this equation is just the most compact way to write all possible combinations of derivatives at any order N. for us we want the dirac eqn so $[D\psi]^\alpha = [(i\gamma^\mu \partial_\mu + m)\psi]^\alpha$ spin index not space-time

Why introduce this?

- Need a way to characterize Diff eq

- introduce the symbol of a diff. op. D (in this case Principal symbol)

$$\sigma(D, \xi) = \sum_{|M| = N} A^{M\alpha}{}_a \xi_M = P$$

- What is it?

- Associates a Polynomial to a diff. eq. Properties of Polynomial teach us about D.

- in Particular if $P \neq 0$ for all $x \in M$; $\xi \in \mathbb{R}^m - \{0\}$; $k = k'$; $\dim E = \dim F$ Then we say D is elliptic

- How do I compute σ?

i) easy just construct a vector $(\xi_1 \cdots \xi_m)$

m - dimension of M.

2.) follow recipe

$$\frac{\partial}{\partial x^\mu} \to \xi_\mu \quad \text{i.e.} \quad \frac{\partial}{\partial x^1} \to \xi_1$$

$$\frac{\partial^2}{\partial x^\mu \partial x_\nu} \to \xi_\mu \xi_\nu \quad \text{i.e.} \quad \frac{\partial^2}{\partial x^1 \partial x^2} \to \xi_1 \xi_2$$

$$\frac{\partial^2}{\partial x^\mu \partial x_\mu} \to \xi_\mu \xi_\mu \quad \text{i.e.} \quad \frac{\partial^2}{\partial x^1 \partial x_1} = \frac{\partial^2}{(\partial x^1)^2} \to \xi_1^2$$

Ex.)

Suppose $M = \mathbb{R}^m$ and E & F are line bundles

The Laplacian $\Delta : \Gamma(\mathbb{R}^m, E) \to \Gamma(\mathbb{R}^m, F)$

$$\Delta \equiv \frac{\partial^2}{(\partial x^1)^2} + \cdots + \frac{\partial^2}{(\partial x^m)^2}$$

symmetry Group? $Gl(1, \mathbb{R})$

has symbol

$$\sigma(\Delta, \xi) = \xi_1^2 + \cdots \xi_m^2 = \sum_\mu (\xi_\mu)^2 \equiv P_\Delta$$

We see P_Δ is nowhere zero for $\xi \in \mathbb{R}^m - \{0\}$

$\therefore$ Δ is elliptic.

Ex.) Same setup but let fiber metric on TM be $\eta = \text{diag}(-, + \cdots\cdots)$

Then the d'Alembertian $\Box = \partial_\mu \partial^\mu$

$$\Box = \partial_\mu \partial^\mu = -\left(\frac{\partial}{\partial x^0}\right)^2 + \left(\frac{\partial}{\partial x^1}\right)^2 + \cdots \left(\frac{\partial}{\partial x^{m-1}}\right)^2$$

has symbol

$$\sigma(\Box, \xi) = -\xi_0^2 + \xi_1^2 + \cdots \xi_m^2 \equiv P_\Box$$

and is not everywhere nonzero since

$$\xi_\mu' = \begin{cases} \xi_1^2 + \cdots \xi_m^2 & , \mu = 0 \\ \xi_\mu & , \mu = 1, \cdots m-1 \end{cases}$$

makes $P_\Box = 0$ $\therefore$ $\Box$ is not elliptic.

(This says that $P_\Box$ vanishes on the lightcone)

We have a measure if D is elliptic
- What is Fredholm?

Def $\triangleq$ An elliptic diff. op. (linear)
whose kernel $\ker D = \{ s \in \Gamma(M, E) \mid Ds = 0 \}$
and cokernel $\operatorname{coker} D = \Gamma(M, F) / \operatorname{Im} D$
are finite is a Fredholm operator.

Just as before define inner products on E & F

$E: \langle \, , \, \rangle_E$ & $F: \langle \, , \, \rangle_F$

Then the adjoint $D^+ : \Gamma(M, F) \to \Gamma(M, E)$ is
defined as
$$\langle s', Ds \rangle_F = \langle D^+ s', s \rangle_E, \quad s \in \Gamma(M, E), \; s' \in \Gamma(M, F)$$

Furthermore $\operatorname{coker} D \cong \ker D^+ = \{ s' \in \Gamma(M, F) \mid D^+ s' = 0 \}$

Proof: for interested reader show bijection $\operatorname{coker} D \to \ker D^+$
let $[s] = \{ s' \in \Gamma(M, F) \mid s' = s + Du, \; u \in \Gamma(M, E) \}$
For any $[s]$ should be some $s_0 \in \ker D^+$. Put $s_0 = s - D \frac{1}{D^+ D} D^+ s$
$\Rightarrow$ $D^+ s_0 = D^+ s - D^+ D (D^+ D)^{-1} D^+ s = D^+ s - D^+ s = 0$ ✓ onto
1-1 Suppose $[s_0] \neq [s_0']$ ($s_0, s_0' \in \ker D^+$) ($[s_0] = [s_0']$ implies $s_0 = s_0'$)
Then $s_0 - s_0' = Du$ $\quad 0 = \langle D^+(s_0 - s_0'), u \rangle_E = \langle s_0 - s_0', Du \rangle_F = \langle Du, Du \rangle_F \geq 0$
Can only be zero for $Du = 0$ Therefore $s_0 = s_0'$ ✓ ; $[s_0] = [s_0']$
$\therefore$ $\ker D^+ \cong \operatorname{coker} D$

- Elliptic ops on compact M are Fredholm.
- For Fredholm ops. The analytic index is expressed
$$\operatorname{ind} D = \dim(\ker D) - \dim(\ker D^+)$$

Spin Complexes

A complex is a set of spaces & maps

$$\xrightarrow{D_{i-2}} E_{i-1} \xrightarrow{D_{i-1}} E_i \xrightarrow{D_i} E_{i+1} \xrightarrow{D_{i+1}} E_{i+2} \to \ldots$$

With D_i nilpotent $\quad D_i \circ D_{i-1} = 0$

(The set of forms $\Omega(M)$ & $D_i = d$ forms the deRham complex and maybe familiar)

For us we will be interested a spin bundle over M ($\dim M = m$) and denote sections of this bundle as $\Delta(M) = \Gamma(M, S(M))$. $m = 2l$

The Spin group $\mathrm{Spin}(M)$ is generated by m dirac matrices $\{\gamma^\alpha\}$

$$\gamma^{\alpha +} = \gamma^\alpha$$

$$\&\quad \{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta} I$$

(We are using Euclidean signature)

The Clifford Algebra by

$$1,\ \gamma^{\alpha_1},\ \gamma^{\alpha_1}\gamma^{\alpha_2} \ldots$$

$$\gamma^{\alpha_1} \ldots \gamma^{\alpha_k} \quad (\alpha_1 < \ldots < \alpha_k)$$

$$\&\quad \gamma^1 \ldots \gamma^{2l} \quad *$$

with $*$ defined $\gamma^{m+1} = i^l \gamma^1 \ldots \gamma^{2l}$

This is chosen so that $(\gamma^{m+1})^2 = I$ & $(\gamma^{m+1})^+ = \gamma^{m+1}$

and with representation so that

$$\gamma^{m+1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

in 4-d ie $m = 4$

Pauli matrices

$$\gamma^\beta = \begin{pmatrix} 0 & i\alpha^\beta \\ -i\bar{\alpha}^\beta & 0 \end{pmatrix} \qquad \alpha^\beta = (I, -i\vec{\sigma})$$

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

- A Dirac spinor $\psi \in \Delta(M)$ (i.e. a section of the spin bundle)
  - irrep of Clifford algebra
  - reducible rep of $\mathrm{spin}(M)$

Irreps of spin(M)

- Seperate $\Delta(M)$ by eigenvalues of γ^{m+1}

$\Rightarrow \gamma^{m+1} = \begin{pmatrix} I & \\ & -I \end{pmatrix} \Rightarrow (\gamma^{m+1})^2 = I$

- eigenvalues ± 1 There are two invariant subspaces Δ^+ & Δ^- and $\Delta = \Delta^+ \oplus \Delta^-$
- $\gamma^{m+1}\psi^\pm = \pm\psi^\pm$, $\psi^\pm \in \Delta^\pm$
- Projections $P^\pm = \frac{1}{2}(I \pm \gamma^{m+1})$

$P^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $P^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

- Clearly for $\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \in \Delta^+ \oplus \Delta^-$ ← direct sum bundle

$P^+\psi = \psi^+$, $P^-\psi = \psi^-$

&

$P^+ + P^- = \frac{1}{2}(I + \gamma^{m+1}) + \frac{1}{2}(I - \gamma^{m+1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$(P^\pm)^2 = P^\pm$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$P^+P^- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

&

$P^\pm P^\mp \psi = 0$

is Dirac operator Elliptic?

- flat space $i\partial\!\!\!/\psi = i\gamma^\mu\partial_\mu\psi$, $\psi \in \Delta(M)$
- Curved space $i\nabla\!\!\!\!/\,\psi = i(\gamma^\alpha e_\alpha{}^\mu\partial_\mu + \gamma^\alpha e_\alpha{}^\mu\omega_\mu)\psi$

$e_\alpha{}^\mu$ – Lorentz frame, $\omega_\mu = \frac{1}{2}i\,\omega_\mu{}^{\alpha\beta}\Sigma_{\alpha\beta}$ ← spin connection

- Think of this like $\nabla \Rightarrow \nabla + A$ for a gauge theory
- What is $\sigma(i\partial\!\!\!/, \xi)$? i.e. What is the symbol of Dirac op.

As before replace derivatives $i\gamma^\mu\partial_\mu = i(\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3)$

$\Rightarrow \sigma(i\partial\!\!\!/, \xi) = i(\gamma^0\xi_0 + \gamma^1\xi_1 + \gamma^2\xi_2 + \gamma^3\xi_3) = i\gamma^\mu\xi_\mu = i\xi\!\!\!/ = P\!\!\!/$

- is $P\!\!\!/ = 0$ for any $\xi \in \mathbb{R}^m - \{0\}$?

$i\xi\!\!\!/ \stackrel{?}{=} 0 \Rightarrow i\xi\!\!\!/\, i\xi\!\!\!/ \stackrel{?}{=} 0$

$-\xi\!\!\!/\xi\!\!\!/ \stackrel{?}{=} 0$

$-\gamma^\mu \xi_\mu \gamma^\alpha \xi_\alpha \stackrel{?}{=} 0$

$\gamma^\mu \gamma^\alpha \xi_\mu \xi_\alpha$

$\begin{pmatrix} 0 & -i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

do it the other way $\Rightarrow \gamma^\alpha \gamma^\mu \xi_\mu \xi_\alpha$, recall $\{\gamma^\mu, \gamma^\alpha\} = 2\delta^{\mu\alpha} I$

add

$\frac{1}{2}(\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu)\xi_\mu\xi_\alpha$

$\delta^{\mu\alpha}\xi_\mu\xi_\alpha = \xi_\mu\xi^\mu$

or $\mu \neq \alpha$ $\gamma^\mu\gamma^\alpha = -\gamma^\alpha\gamma^\mu$

$\mu = \alpha$ $2\gamma^\mu\gamma^\alpha = 2I$

$\gamma^{\mu 2} = I$

We showed earlier this is everywhere nonzero for $\xi_n \in \mathbb{R}^m - \{0\}$

Therefore $i\xi_n\gamma^\mu$ is nonzero everywhere

$\therefore$ $i\nabla\!\!\!\!/$ is Elliptic.

We then expand $i\nabla\!\!\!\!/ = i\gamma^\mu(\partial_\mu + \omega_\mu)$

$i\begin{pmatrix} 0 & i\alpha^\mu \\ -i\bar{\alpha}^\mu & 0 \end{pmatrix}(\partial_\mu + \omega_\mu)$

$= \begin{pmatrix} 0 & -\alpha^\mu(\partial_\mu + \omega_\mu) \\ \bar{\alpha}^\mu(\partial_\mu + \omega_\mu) & 0 \end{pmatrix} = \begin{pmatrix} 0 & D^+ \\ D & 0 \end{pmatrix}$

Therefore D^+ is the adjoint of D since

if $\psi^+ \in \Delta^+$; $\psi = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}$

$i\nabla\!\!\!\!/\psi^+ = \begin{pmatrix} 0 & D^+ \\ D & 0 \end{pmatrix}\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ D\psi^+ \end{pmatrix} \in \Delta^-$

and $\psi = \begin{pmatrix} 0 \\ \psi^- \end{pmatrix}$ $\psi^- \in \Delta^-$

$i\nabla\!\!\!\!/\psi^- = \begin{pmatrix} 0 & D^+ \\ D & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \psi^- \end{pmatrix} = \begin{pmatrix} D^+\psi^- \\ 0 \end{pmatrix} \in \Delta^+$

Hence $D = i\nabla\!\!\!\!/ P^+ : \Delta^+ \to \Delta^-$; $D^+ = i\nabla\!\!\!\!/ P^- : \Delta^- \to \Delta^+$

and we have a two term complex

$$\Delta^- \underset{D}{\overset{D^+}{\rightleftarrows}} \Delta^+$$ The spin complex!

The Analtic index

$$\mathrm{ind}\, D = \dim(\ker D) - \dim(\ker D^{+})$$
$$= \nu^{+} - \nu^{-}$$

# of 0-eigenmodes of Positive Chirality

# of 0 eigenmodes With negative Chirality