HEP Seminar: A SUSY proof of the Dirac index theorem
Exercises Week 2
Casey Cartwright September 24, 2017

Since the topic of index theorems contains many pieces of mathematical vocabulary here are some practice problems to help you learn all of the words I used in seminar.

1)What is the polynoimal symbol associated with the following operators,

1. $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0$
2. $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 f}{\mathrm{d}y^2} = 0$
3. $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 f}{\mathrm{d}y^2} + 8\frac{\mathrm{d}^2 f}{\mathrm{d}y\mathrm{d}x} = 0$

Are these operators elliptic? Additionally show that the following differential operator defined via the symbol

$$\sigma(\nabla, \xi) = A_{11}\xi_1^2 + A_{21}\xi_1\xi_2 + A_{22}\xi_2^2 \tag{1}$$

is elliptic if and only if the symbol σ is an ellipse in ξ space. I'll remind you that to show an if and only if statement you must show both directions. That is suppose x is y if and only if y is z means suppose x is y show that y is z, and suppose y is z show x is y.

2) The special linear group is defined

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{C} \text{ and } ab - bc = 1 \right\} \tag{2}$$

Notice that this is not the special unitary group $SU(2)$ in fact $SU(2) \subset SL(2,\mathbb{C})$. Additionally let $SO(1,3)^{\uparrow}$ be the special orthogonal group with $\Lambda^0_0 > 0$ i.e. the proper othochronous Lorentz group. It turns out that the special linear group is a double cover of the proper othochronous Lorentz group. This is just what we did in group theory this last week. So you know the drill, given that the homomorphism is $\Lambda : SL(2,\mathbb{C}) \to SO(1,3)^{\uparrow}$ is given by $\Lambda(g)X = gXg^{\dagger}$ show that the action $\Lambda(g)X$ with,

$$g = \begin{pmatrix} \cosh\alpha/2 + \sinh\alpha/2 & 0 \\ 0 & \cosh\alpha/2 - \sinh\alpha/2 \end{pmatrix} \tag{3}$$

is the same as a boost around the z-axis. The velocity will be $\nu = \tanh\alpha$. Hints: write $\Lambda(g)X = x' = x'^{\mu}\sigma_{\mu}$ with $\sigma_{\mu} = (I, \vec{\sigma})$ and call x'^{μ} the transformed vector. Then write $X(x) = x^{\mu}\sigma_{\mu}$ and compute $gXg^{\dagger}$ for the given element. Then $x'^{\mu}\sigma_{\mu} = gX(x)g^{\dagger}$.

3) In preperation for next week consider the following action,

$$\int_0^{\beta} \mathrm{d}t \left(\frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{1}{2} g_{\mu\nu}\psi^{\mu}\frac{D\psi^{\nu}}{Dt} \right) \tag{4}$$

Here $\frac{D\psi^\nu}{Dt} = \frac{\mathrm{d}\psi^\nu}{\mathrm{d}t} + \dot{x}^\lambda \Gamma^\nu_{\lambda\kappa}\psi^\kappa$ is the covariant derivative along the curve $x(t)$. Calculate the equations of motion for ψ and for x. Assuming that ψ is a grassman number (i.e. $\theta\phi = -\phi\theta$ for grassman numbers ϕ and θ) and x is a classical (commuting) number.

4) Speaking of grassman numbers have you ever calcuated the intgral of a grassman number before? If not this problem will fix you right up. These two formulas will be good to know.

$$\int \mathrm{d}\theta = 0, \qquad \int \mathrm{d}\theta\theta = 1 \tag{5}$$

Thats right. To calcuate the value you take the derivative of the integrand. With this in mind the fermionic harmonic ossicillator is desribed by grassman numbers. The Hilbert space is spanned by $\mathcal{H} = \mathrm{span}\{|0>, |1>\}$. Knowing that $\theta^2 = 0$ and $|\theta >= |0> + |1> \theta$, show that the completeness relation can be written,

$$\int \mathrm{d}\theta^* \mathrm{d}\theta |\theta >< \theta| e^{-\theta^*\theta} = I \tag{6}$$