HEP Seminar: A SUSY proof of the Dirac index theorem
Exercises Week 1

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Since the topic of index theorems contains many pieces of mathematical vocabulary here are some practice problems to help you learn all of the words I used in seminar.

1) Construct a vector bundle $E \to S^1$ using $S^1 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ as a base space (i. e. $S^1 = M$) and the interval $[-1,1] = F$ as the fiber space. The construction should include definitions for all necessary maps, that is, explicitly write down π, the trivializations ϕ, the transition functions, what is your open cover on S^1 etc. You should find that you get two different bundles, what are they? How do you construct one vs the other?

2) Let E_1 and E_2 denote the two distinct bundles you created above. Construct the direct (Whitney) sum bundles $E_1 \oplus E_1$ and $E_2 \oplus E_2$. Show that both are trivial bundles. In seminar I gave one definition of triviality of a bundle, but there are actually many definitions. Feel free to look at M. Nakahara's text or another to determine what the best definition of triviality is for this problem.

Bonus! Another example of a fiber bundle is called a principal fiber bundle. A principal bundle has all of the same definitions as vector bundles however the fiber space F is identical to group space G. Since $F = G$, in addition to a left group action, a right group action can be introduced as $R_g u = ug$ for $g \in G, u \in P$. Construct the bundle $P(S^2, U(1))$, a principal bundle with the sphere S^2 as a base space and $U(1)$ as the fiber space. Show that the 3-sphere S^3 is diffeomorphic to the constructed principal bundle $P(S^2, U(1)) \cong S^3$. (This is essentially what we did as a group theory homework problem this last week, its just seen from a different perspective)