Graduate Qualifying Exam

Department of Physics & Astronomy, University of Alabama

6-7 January 2014

General Instructions

- No reference materials are allowed.
- Do *all* your work in the corresponding answer booklet.
- On the cover of each answer booklet, make sure to write your *assigned number* and the part number/subject.
  Exams are graded anonymously, so do not write your name.
- Turn in the question sheet for each part with the answer booklet.
- 120 minutes are allotted for each part, except for Thermal Physics (60 minutes).
- **Calculator policy**: Use of a graphing or scientific calculator is permitted provided that it has *none* of the following capabilities:
  - programmable
  - algebraic operations
  - storage of ASCII data

  Handheld computers, PDAs, and cellphones are explicitly prohibited.

Part I: Electricity and Magnetism

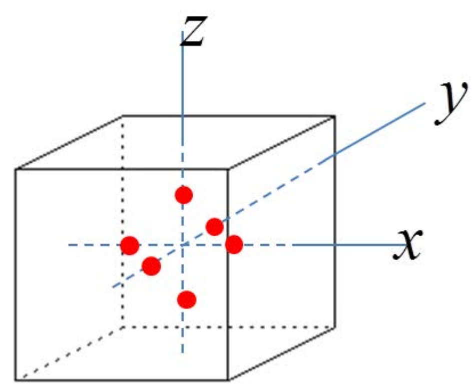
Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

Useful vector formulas:

$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$, where $\vec{A}$ is a vector field

Gradient in spherical coordinates: $\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

1. A box in the shape of a cube is centered at the origin with faces normal to the Cartesian (x,y,z) directions. The length of the side of the box is $4a$. Electrical charges, q, are placed at the following 6 positions: $(x = \pm a, y = 0, z = 0\ )$; $(x = 0, y = \pm a, z = 0\ )$; $(x = 0, y = 0, z = \pm a)$. What is the average value of the x-component of the electric field over the face of the cube at $x = 2a$?

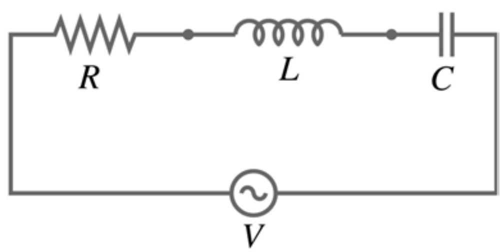


2. Consider Maxwell's equations applied to a non-conducting, homogeneous, isotropic, linear medium without sources.
   (a) Show that both $\vec{E}$ and $\vec{H}$ satisfy the homogeneous Helmholtz (wave) equation. Justify each step.
   (b) If an oscillating electric field, $\vec{E} = E_0\hat{x}\cos(kz - \omega t)$, exists in the medium specified above, what is the corresponding magnetic field, $\vec{B}$?
   (c) Verify that all Maxwell's equations are satisfied by these $\vec{E}$ and $\vec{B}$.

3. A sphere of radius R has uniform polarization $\vec{P}$. The polarization is in the z-direction.
   (a) Find the surface charge density.
   (b) Find the potential inside and outside the sphere.
   (c) Find the electric field inside and outside the sphere.
   (d) Find the displacement field inside and outside the sphere.

   (Hint: the surface charge density causes a discontinuity in the normal component of the electric field that can be determined from Gauss law.)

4. Consider the series RLC circuit shown in the figure.



   (a) Express the voltages $V_R(t)$, $V_L(t)$ and $V_C(t)$ across the resistor, inductor and capacitor, respectively, in terms of the charge, $Q(t)$, on the capacitor and its derivatives.
   (b) The source provides an alternate voltage $V(t) = Re(V_0 e^{i\omega t})$. Write down a differential equation for $Q(t)$ or the current, $I(t)$, and solve for the current. Express the current in terms of V_0, R, L, C and ω.
   (c) Calculate the resonance frequency of the circuit.

5. A long straight cylindrical wire of length L, radius r_w and resistance R carries a constant current I.

   (a) Evaluate Poynting's vector at the surface of the wire.

   (b) Compute the energy flow using the Poynting's vector and compare it to the energy dissipated in heat.

6. A planar interface seperates the vacuum from a medium with permittivity $\epsilon = 4\epsilon_0$ and permeability $\mu = \mu_0$. The interface is at the location defined by $z = 0$, while the medium occupies the space $z > 0$. An electromagnetic wave traveling from the vacuum is normally incident to the interface. The electric and magnetic fields of the incident wave are, respectively, $\vec{E}(\vec{r}, t) = \hat{x} E_i \cos(kz - \omega t)$ and $\vec{B}(\vec{r}, t) = \hat{y} B_i \cos(kz - \omega t)$.

   (a) Use one or more of Maxwell's equations to obtain B_i in terms of E_i.

   (b) Use Poynting's vector to calculate the time averaged power per unit area in the incident beam.

   (c) Obtain the boundary conditions that connect the amplitudes of the incident, reflected, and transmitted waves. Draw diagrams if needed.

   (d) What fraction of the incident power is reflected?

Part II: Quantum Mechanics

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
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1. Consider a system comprised of two angular momenta labeled "A" and "B", respectively. The total angular momentum quantum number of A is 1 and the total angular momentum quantum number of B is $\frac{3}{2}$.

   (a) If the total angular momentum of the system were measured, what would be the possible outcomes? Give your answers in units of $\hbar$. Justify your answer.

   (b) Suppose that the system is in a state where the total angular momentum quantum number of the system is $\frac{5}{2}$ and the azimuthal quantum number is $\frac{3}{2}$. What is the probability that a measurement of the z-component of the B's angular momentum would yield the result $\frac{1}{2}\hbar$? Note that the angular momentum raising and lowering operators are given by

   $$J_\pm \mid j, m >= \hbar\sqrt{j(j+1) - m(m \pm 1)} \mid j, m \pm 1 >$$

   where j is the total angular momentum quantum number and m is the azimuthal angular momentum quantum number.

2. (a) Suppose you have three particles and they can be in three distinct one–particle states with associated eigenfunctions $\psi_a(x)$, $\psi_b(x)$ and $\psi_c(x)$. How many different three–particle states can be constructed if (i) they are distinguishable particles, (ii) they are identical bosons, and (iii) they are identical fermions?

   (b) Assuming that there is a large number of particles in thermal equilibrium at temperature T with chemical potential μ. Write down the expression giving the most probable number of particles occupying a particular state with energy E for each of the following cases: (i) distinguishable particles, (ii) identical fermions, and (iii) identical bosons.

   (c) Assume a system in which the allowed energies E_i, $i = 1, 2, 3, 4...$, are discrete and non–degenerate. The total energy of the system is E, the total number of particles in the system is N, and the system is in thermal equilibrium. Assume that N is very large. For the case of identical fermions, write down the two equations from which one could determine the temperature T and chemical potential μ of the system in terms of E and N.

3. Electromagnetic transitions between atomic states can be described in terms of matrix elements of the electric dipole operator if the wavelength of the radiation emitted or absorbed is very long compared to the characteristic size of the atom. Suppose that a hydrogen atom decays from the state (n,ℓ,m) to the state (n$'$,ℓ',m$'$). *Derive* the selection rule relating m and m$'$ and *state* the selection rule relating ℓ' and ℓ.
Hint: write down the matrix element of the dipole operators and use the following identities:

$$[L_z, x] = i\hbar y \quad [L_z, y] = -i\hbar x \quad [L_z, z] = 0$$

4. For any two observables A and B, associated with the operators $\hat{A}$ and $\hat{B}$, respectively, the generalized statistical interpretation of quantum mechanics gives

$$\sigma_A \sigma_B \geq \frac{1}{2} \, |\langle [\hat{A}, \hat{B}] \rangle|,$$

where σ_A (σ_B) represents the standard deviation from the mean of a measurement of A (B) while the angle brackets indicate the expectation value for some state. For any operator $\hat{A}$, define the uncertainty, Δt, of the time measurement by this equation:

$$\sigma_A = \left| \frac{d \, \langle \hat{A} \rangle}{dt} \right| \Delta t.$$

Obtain the energy-time uncertainty principle from these relationships and state how it is to be interpreted.

5. Consider a particle of mass m moving in an isotropic three-dimensional harmonic oscillator potential,

$$V(r) = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2 + z^2).$$

(a) Find the allowed energies for this system. Give the degeneracy of each of the three-lowest allowed energies. In answering this question, you may take as given the results for the one-dimensional harmonic oscillator, but you must explicitly show the connection between the 3–d case and the 1–d case.

(b) The time–independent wave function for the ground state is

$$\psi(\vec{r}) = Ae^{-\alpha r^2}$$

where A is a constant and

$$\alpha \equiv \frac{m\omega}{2\hbar},$$

with

$$\omega \equiv \sqrt{\frac{k}{m}}.$$

Find the expectation value of r for this state.

(c) What is the angular momentum of the ground state? Explain your answer.

Useful integrals:

$$\int_0^\infty dx\ e^{-x^2} = \frac{\sqrt{\pi}}{2}\ , \quad \int_0^\infty dx\ x^{2n} e^{-x^2} = \frac{(2n-1)!!}{2^n}\frac{\sqrt{\pi}}{2}\ , \quad \int_0^\infty dx\ x^{2n+1} e^{-x^2} = \frac{n!}{2}\ ,$$

where n is an integer and $(2n-1)!! \equiv (2n-1)\cdot(2n-3)\cdot(2n-5)\cdot \ldots\ldots \cdot 1$.

6. A particle moving in one dimension is confined to an infinite square well of width L. The well is located at $0 < x < L$. At time $t = 0$, the wave function is given by

$$\Psi(x, 0) = Ax(L - x)$$

within the interval $0 < x < L$ and is zero elsewhere.

(a) Determine A to correctly normalize the wavefunction.

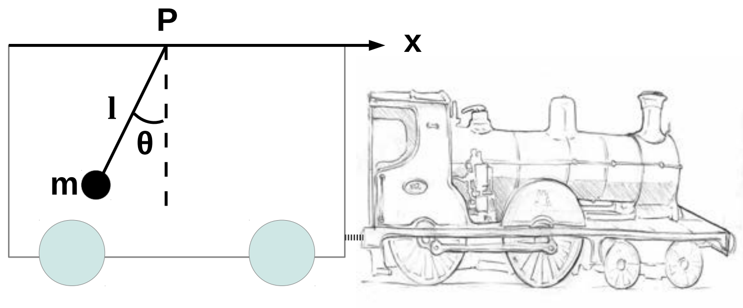
(b) What is the probability that a measurement of the particle's position at $t = 0$ yields a result in the range $\frac{L}{4} < x < \frac{3L}{4}$?

(c) Find the expectation value for the energy.

Part IIIa: Classical Mechanics

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

1. A simple pendulum consisting of a mass m hung from a thin, massless cord of length l is suspended from a point P in a railroad car that is accelerating in the horizontal direction with a constant acceleration $a > 0$.



   (a) Defining θ to be the angle the pendulum is displaced from the vertical position, write down the Lagrangian $L(\theta, \dot{\theta}, t)$ for the pendulum. Assume $x(0) = 0$ and $\dot{x}(0) = 0$, where x denotes the horizontal position of P as measured in an inertial frame.

   (b) Find the equation of motion for θ in terms of g, l, and a.

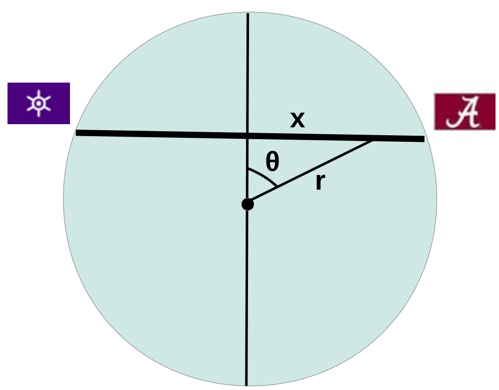
   (c) Finally, find the equilibrium angle of the pendulum and the frequency of small oscillations about this angle.

2. Assume that the Earth is a perfect sphere and has a constant density.

   (a) Sketch the magnitude of the gravitational force as a function of distance from the center of the Earth, for distances both less than and greater than the radius of the Earth, R_E.

   (b) A straight tunnel is bored through the Earth between Tuscaloosa and Tokyo, passing 2000 km from the center of the Earth, as shown in the figure below. A package of mass m is dropped into the tunnel on the Tuscaloosa side. If x measures the distance of the package from the midpoint of the tunnel, write down the equation of motion of the package in terms of x, g, and R_E. Ignore all forms of friction.

   (c) How many minutes will it take the package to reach the other end of the tunnel? You can use, $R_E = 6400$ km and $g = 10$ m/s^2.



3. A particle of mass m experiences a central force given by $F(r) = -kr$, where k is a positive constant of appropriate dimensions.

   (a) What is the effective one dimensional potential in terms of m, k, and the angular momentum ℓ?

   (b) For a given angular momentum $\ell > 0$, there is one circular orbit. Find its radius and show that it is stable under perturbation of the radial coordinate.

   (c) Find the frequency of small oscillations in r about the stable orbit.

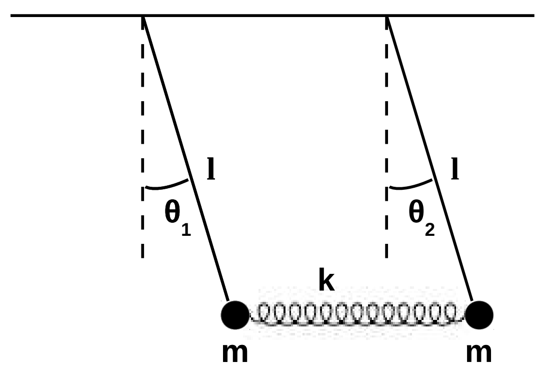
4. Consider a particle of mass m subjected to an attractive central force.

   (a) Prove that the orbit of the particle is confined to a plane.

   (b) Write down the Lagrangian of the particle for a general central potential $U(r)$, and find the equation of motion for the radial coordinate r in terms of m, $U(r)$, and the angular momentum ℓ.

   (c) For some specific values of the angular momentum ℓ, show that the spiral trajectories described by $r(\theta) = r_0 e^{\theta}$ result from an inverse cube force $F(r) = -\frac{k}{r^3}$ and find ℓ in terms of m and k.

5. The bobs of two identical simple pendulums are connected by a spring. The mass of each bob is m, the length of each pendulum is l, and the spring has a spring constant k. The spacing between the pivots of the pendulums is equal to the the unstretched length of the spring. Assume that the angular displacements are small.



   (a) Find the frequencies, ω_1 and ω_2, of the two normal modes.

   (b) Find the two eigenmodes.

   (c) At $t = 0$, one bob is displaced by a small angle, $\theta_1(0) = a$, while the other remains at zero, $\theta_2(0) = 0$. Both are initially at rest. Find $\theta_1(t)$ and $\theta_2(t)$ for each bob in terms of a, ω_1 and ω_2.

6. A particle of mass m is subjected to the following force:

$$F(v) = a - bv$$

   where a and b are positive constants with appropriate units, and v is the magnitude of the velocity.

   (a) Find an expression for the velocity of the particle as a function of time in terms of a, b, m, and the initial velocity $v_0 \geq 0$. Express the velocity in the limit $t \to \infty$ in terms of these same quantities.

   (b) Find an expression for the position of the particle as a function of time in terms of a, b, m, v_0, and the initial position x_0.

   (c) If the particle starts at rest, how far does it travel before its velocity is one half its maximum velocity?

Part IIIb: Thermodynamics

Do any 2 of the 3 problems.
If you try all 3 problems, indicate clearly which 2 you want marked.
If there is no clear indication, the first 2 problems will be marked.

Useful constants:
- The gas constant: $R = 8.31$ J/(mol K)
- Boltzmann constant: $k = 1.38 \times 10^{-23} J/K$
- Avogadro number $N_A = 6.023 \times 10^{23} mol^{-1}$
- Latent heat of ice: 3.33×10^5 J/kg
- Specific heat of water: 4186 J/(kg K)

1. A rigid tank of small mass contains 1 mol of argon (monoatomic, ideal gas), initially at 180°C and 100 kPa. The tank is placed into a reservoir at 0°C and is allowed to cool to thermal equilibrium. The tank and the reservoir constitute an isolated system.
   (a) Calculate the volume of the tank.
   (b) Calculate the change in internal energy of the argon.
   (c) Calculate the energy transferred by heat to the reservoir.
   (d) Calculate the change in entropy of the argon.
   (e) Calculate the change in entropy of the reservoir.

2. One mole of a monoatomic ideal gas has a volume $V_i = 10$ L and is at pressure $P_i = 10^5$ Pa. This gas is running through the following 3-step process.
   1. At constant temperature, the gas is compressed to one half of its original volume.
   2. The gas is decompressed adiabatically till it reaches its original pressure P_i.
   3. Now, the gas expands at constant pressure till it reaches its original volume V_i.

   (a) Sketch in a diagram P vs V of the 3-step process.
   (b) Calculate how much work is done on the gas in step (1).
   (c) During step (1), was heat transferred into or out of the gas? Explain your answer.
   (d) Calculate the final volume after completion of step (2).
   (e) Calculate how much work is done by the gas in step (3).
   (f) During step (3), does heat flow into or out of the gas? Explain your answer.

3. A 10 g ice cube (originally at 0°C) is placed in 100 g of water (originally at 20°C).
   (a) Calculate the final temperature of the system.
   (b) Calculate the change in entropy of the system assuming that it is an isolated system.