Qualifying Examination

January 11–13, 2012

General Instructions

- No reference materials are allowed.
- Do *all* your work in the corresponding answer booklet.
- On the cover of each answer booklet put only your *assigned number* and the part number/subject.
- Turn in the questions for each part with the answer booklet.
- 120 minutes are alloted for each part, except for Thermal Physics (60 minutes).
- **Calculator policy**:
  Use of a graphing or scientific calculator is permitted provided that it does not have any of the following capabilities:

  – programmable
  – algebraic operations
  – storage of ASCII data

  Handheld computers, PDAs, and cell phones are explicitly prohibited.

Part I: Electricity and Magnetism

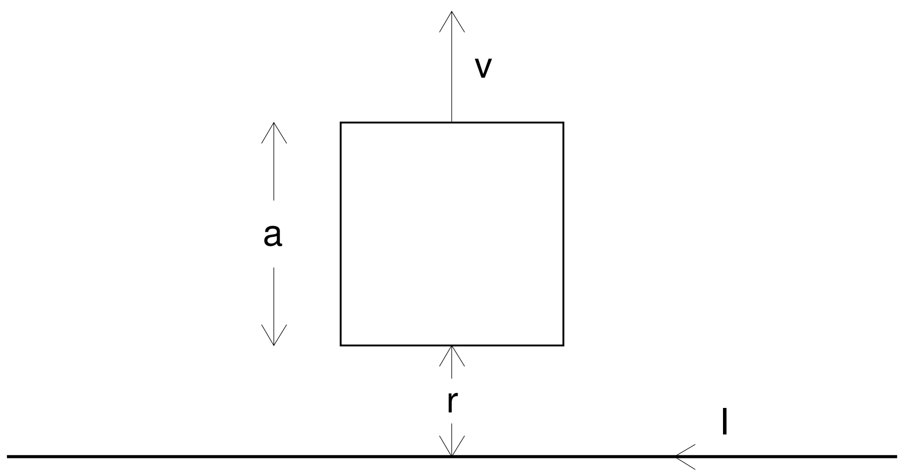
Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which ones you want marked.

Problem 1. Consider a spherical vacuum capacitor consisting of inner and outer thin conducting spherical shells whith charge $+Q$ on the inner shell of radius R_a and $-Q$ on the outer shell of radius R_b.

(a) What is the electric field $\mathbf{E}$ everywhere in space as a function of the distance r from the center of the spherical conductor?

(b) What is the capacitance of this capacitor?

Problem 2. A square loop with height and width a moves with a constant velocity $\mathbf{v}$ away from an infinitely long straight wire carrying a current I in the plane of the loop.



(a) What is the magnetic flux through the loop when the lower side is at a distance r away from the wire, as shown in the figure.

(b) For the situation described in part (a) find the current induced in the loop (magnitude and direction).

Problem 3. An inductor with inductance L and a capacitor with capacitance C are connected in series. The current in the circuit is forced to increase linearly in time, i.e., $I(t) = kt$. The capacitor initially has no charge.

(a) Determine the voltage across the inductor as a function of time.

(b) Determine the voltage across the capacitor as a function of time.

(c) Determine the time at which the energy stored in the capacitor equals the energy stored in the inductor.

Problem 4. The electric field of an electromagnetic wave is given by

$$\mathbf{E} = E_0 \left[\cos(kz - \omega t) + \cos(kz + \omega t)\right] \mathbf{e}_x,$$

where $\mathbf{e}_x$ is the unit vector in the x direction.

(a) What is the associated magnetic field $\mathbf{B}(x, y, z, t)$?

(b) What is the Poynting vector $\mathbf{S}$ of this wave?

(c) Calculate the time average of the Poynting vector and comment briefly on your result.

Trigonometric identities:

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta,$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta.$$

Problem 5. An infinitely long solid cylinder ($\mu_r = 1$) of radius R carries a constant current density j_0.

(a) Show that the vector potential defined by

$$\mathbf{A}(\mathbf{r}) = A_z(\rho)\mathbf{e}_z$$

with $\mathbf{e}_z$ being the unit vector in the z direction and

$$A_z(\rho) = \begin{cases} -\mu_0 \dfrac{I}{4\pi R^2}\rho^2 & \text{for} \quad \rho < R \\[2ex] C \ln \dfrac{\rho}{R} - \mu_0 \dfrac{I}{4\pi} & \text{for} \quad \rho \geq R \end{cases}$$

satisfies the Poisson equation. In the expression of $A_z(\rho)$ above C is a constant and I is the total current.

(b) Using this vector potential calculate the magnetic field inside and outside the cylinder and determine the value of the constant C.

Note: in cylindrical coordinates the Laplace operator reads:

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2},$$

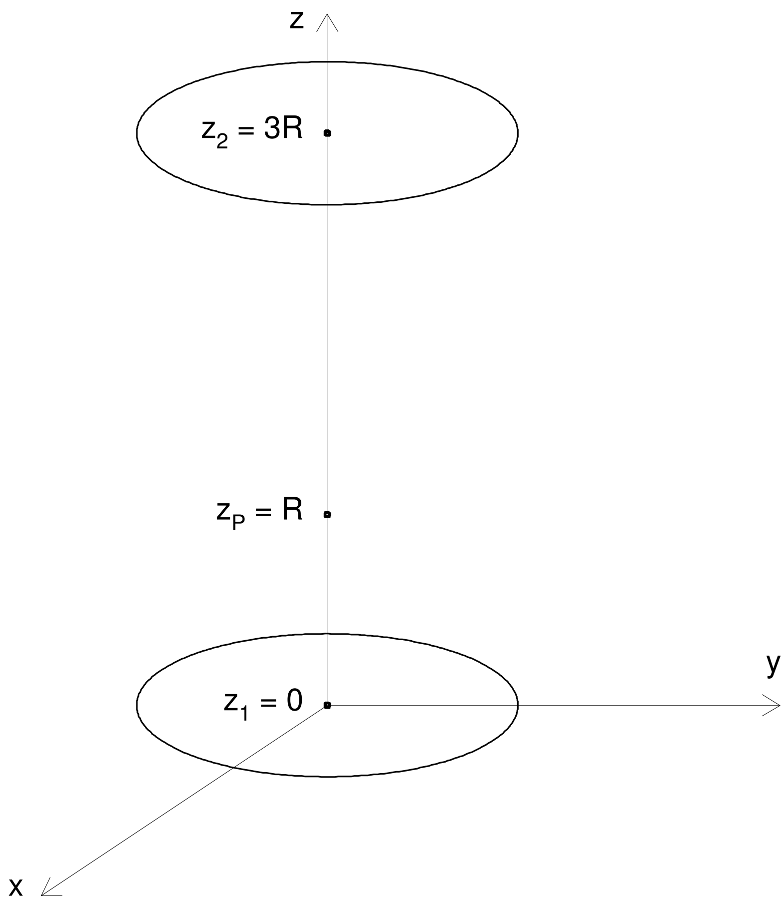
while the curl of any vector field $\mathbf{V}$ is given by

$$\boldsymbol{\nabla} \times \mathbf{V} = \left(\frac{1}{\rho}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)\mathbf{e}_\varphi + \frac{1}{\rho}\left[\frac{\partial(\rho V_\varphi)}{\partial \rho} - \frac{\partial V_\rho}{\partial \varphi}\right]\mathbf{e}_z.$$

Problem 6. A non-conducting ring of radius R and uniform charge q_1 is placed at the center of the coordinate system in the (x, y)-plane.

(a) Calculate the electric field $\mathbf{E}$ at a point z along the z-axis.

(b) A second ring of equal radius and uniform charge q_2 is placed with its center at $z = 3R$, parallel to the first one. The net electric field at a point P along the z-axis at $z = R$ is now identically zero. Calculate the q_1/q_2 ratio.



Part II: Quantum Mechanics

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which ones you want marked.

Problem 1. In a 1-dimensional system, a particle of mass m is trapped in a potential well

$$V(x) = \begin{cases} V_0 > 0 & \text{for} \quad |x| > L \\ 0 & \text{for} \quad |x| < L. \end{cases}$$

When a potential is invariant under $x \leftrightarrow -x$, the wave function of the system is decomposed into even and odd functions of x. Here we consider even functions only.

(a) Write the solutions of the time-independent Schrödinger equation in the regions $|x| < L$ and $|x| > L$ in terms of $k = \sqrt{2mE/\hbar^2}$ and $\lambda = \sqrt{2m(V_0 - E)/\hbar^2}$ for energies $E < V_0$.

(b) Considering the boundary conditions at $|x| = L$, find an equation by which the energy spectrum of the system is determined.

(c) In the limit $V_0 \to \infty$, find the lowest energy of the spectrum.

Problem 2. A particle propagates in the positive x-direction under the influence of a potential

$$V(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ V_0 > 0 & \text{for} \quad x > 0. \end{cases}$$

(a) Determine the wavefunctions in the two regions $x < 0$ and $x > 0$ in terms of $k = \sqrt{2mE/\hbar^2}$ and $\lambda = \sqrt{2m(E - V_0)/\hbar^2}$ for energies $E > V_0$.

(b) Calculate the reflection coefficient (R) and the transmission coefficient (T) in terms of k and λ.

(c) Confirm the relation $T + R = 1$ from the results in part (b) above.

Problem 3. The Hamiltonian of the harmonic oscillator in a 1-dimensional system is given by

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2.$$

We introduce

$$Q = \sqrt{\frac{m\omega}{\hbar}}\, x \quad \text{and} \quad P = -i\,\frac{d}{dQ}$$

satisfying $[Q, P] = i$.

(a) Express H in terms of P and Q.

(b) The annihilation (a) and creation ($a^\dagger$) operators are defined as

$$a = \frac{1}{\sqrt{2}}(Q + iP) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP).$$

Verify that $[a, a^\dagger] = 1$ and $H = \hbar\omega(N + 1/2)$, where $N = a^\dagger a$ is the so-called number operator.

(c) Suppose u_n is the eigenstate of N satisfying $Nu_n = nu_n$. Show that $au_n \propto u_{n-1}$.

(d) Find the wavefunction of the ground state of the system defined as the solution of $au_0 = 0$. Useful integral:

$$\int_{-\infty}^{\infty} dx\, e^{-Ax^2} = \sqrt{\frac{\pi}{A}}.$$

Problem 4. Consider the system with only two states based on the Hamiltonian $H = H_0 + \epsilon H_1$, where

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & -E_0 \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) We first consider the limit of $\epsilon = 0$, in which the Hamiltonian is already diagonalized. Determine the two eigenstates in the form of column vectors and the corresponding eigenvalues.

(b) Assuming $0 < \epsilon \ll E_0$, find the energy eigenvalues in first order perturbation with respect to $\epsilon/E_0 \ll 1$.

(c) In fact, one can easily diagonalize the 2×2 matrix H. Find the exact eigenvalues.

(d) Expand the exact solutions of part (c) with respect to $\epsilon/E_0 \ll 1$ and verify the results obtained in part (b).

Problem 5. Consider a quantum mechanical system with two possible orthogonal states ("flavors" A and B) given by

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad |\psi_B\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In this system, the normalized energy eigenstates are

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the corresponding eigenvalues $E_1 = E_0$ and $E_2 = 2E_0$.

(a) If the wavefunction at time $t = 0$ is $|\psi(0)\rangle = |\psi_A\rangle$, find the wavefunction $|\psi(t)\rangle$ at an arbitrary time t.

(b) The flavor is measured to be A at time $t = 0$, i.e., the same initial condition as in part (a) above. What is the probability for the flavor to be B at a time $t = \pi\hbar/(3E_0)$?

Problem 6. The energy spectrum of the hydrogen atom is determined by the principal quantum number $n = 1, 2, 3, \ldots$ as $E_n = -c_0/n^2$, where c_0 is a constant. (In this problem we do not consider the spin of the electron.)

(a) For $n = 2$, classify the eigenfunctions in terms of quantum numbers corresponding to the orbital angular momentum (ℓ) and the z-component of the the orbital angular momentum (m).

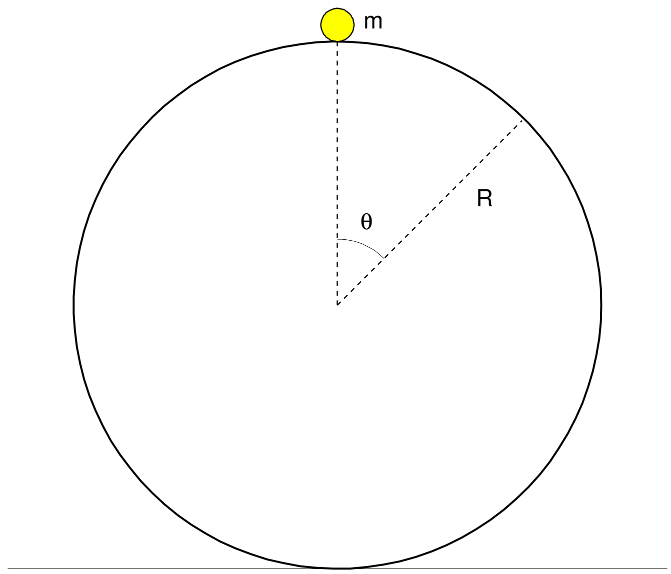
(b) What is the number of independent eigenfunctions for $n = 2$?

(c) What possible values of the orbital angular momentum (initial and final) are allowed for a transition from $n = 2$ to $n = 3$?

Part III: Classical Mechanics

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which ones you want marked.

Problem 1. A point mass m starts from rest from the top of a *stationary* sphere of radius R and slides down the frictionless surface.



(a) At what angle θ (measured with respect to the vertical) does the mass fly off the sphere?

(b) If there is friction between the mass and the sphere, does the mass fly off at a greater or smaller angle as the one found in part (a)? Explain.

Problem 2. A planet of mass m orbits in the gravitational field of the Sun of mass M. Assume that the solar mass is much larger than the mass of the planet, $M \gg m$, so that effectively the Sun can be considered to be at rest.

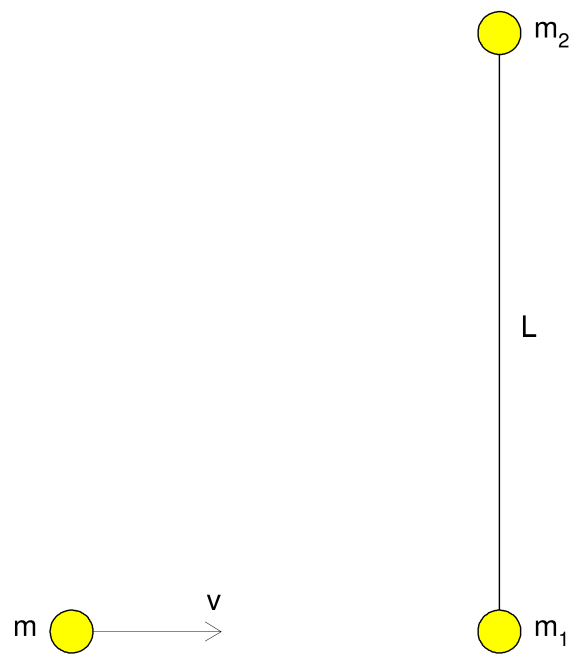
(a) Explain briefly why the orbit of the planet is restricted to a plane.

(b) Determine the Lagrangian for the motion of the planet in polar coordinates, r and θ.

(c) From the Lagrangian in part (b) find the equations of motion.

(d) Show that the azimuthal equation implies a conservation law and identify the conserved quantity. Hence explain qualitatively how the speed of the planet varies with the distance from the Sun along the orbit.

Problem 3. A mass m with velocity $\mathbf{v}$ collides with a system of two masses m_1 and m_2 in *free space*, as shown in the figure below. The two masses m_1 and m_2, initially at rest, are connected with a rigid massless rod of length L. The collision is completely inelastic. The velocity $\mathbf{v}$ before the collision is perpendicular to the rod. Calculate the loss in kinetic energy during the collision and show that it is independent of m_2. Assume that the entire motion takes place in the plane of the figure.



Problem 4. A pendulum of length ℓ having a mass m at the end is subject to a damping force $\mathbf{F} = -2\gamma m\mathbf{v}$, where γ is a constant and $\mathbf{v}$ is the speed.

(a) What is the equation of motion of the pendulum for small angular displacements θ, where θ is the angle with respect to the vertical.

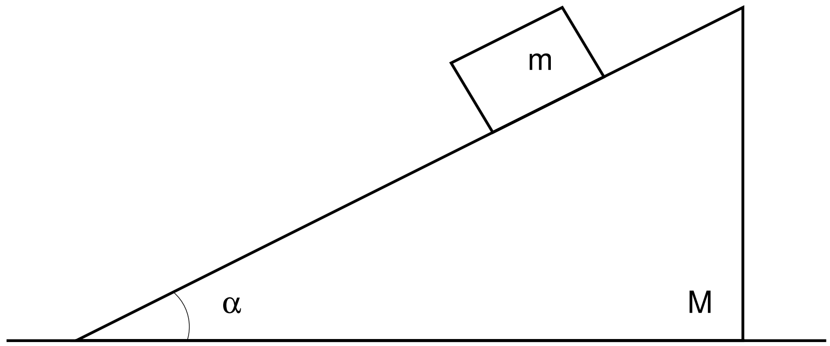
(b) For small values of γ take the solution to be

$$\theta(t) = \theta_0\, e^{-\gamma t}\, \sin \omega t$$

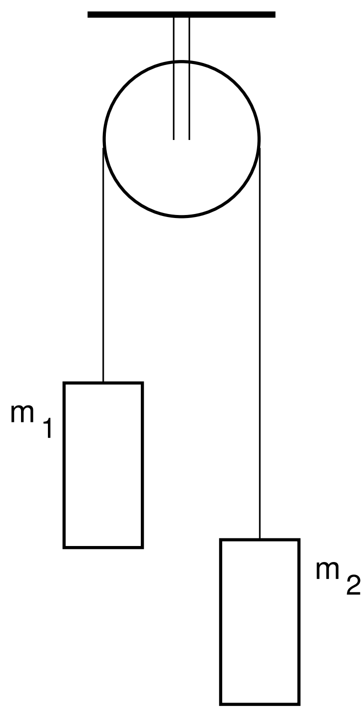
and determine the oscillation frequency ω.

(c) For what values of the damping parameter γ will the pendulum exhibit no oscillatory behaviour?

Problem 5. A block of mass m is released from rest at the top of a wedge of mass M. The wedge rests on a horizontal table and is free to move. All surfaces of contact are assumed to be frictionless. Using the Lagrangian formalism, calculate the acceleration of the wedge.



Problem 6. An Atwood machine consists of two vertically-hanging masses, $m_1 > m_2$, connected by a massless rope thrown over a pulley of moment of inertia I and radius R free to rotate on a frictionless bearing – see figure below. The system is released from rest. Assuming that the pulley turns without slipping, determine the acceleration of the system using either Newtonian or Lagrangian mechanics.



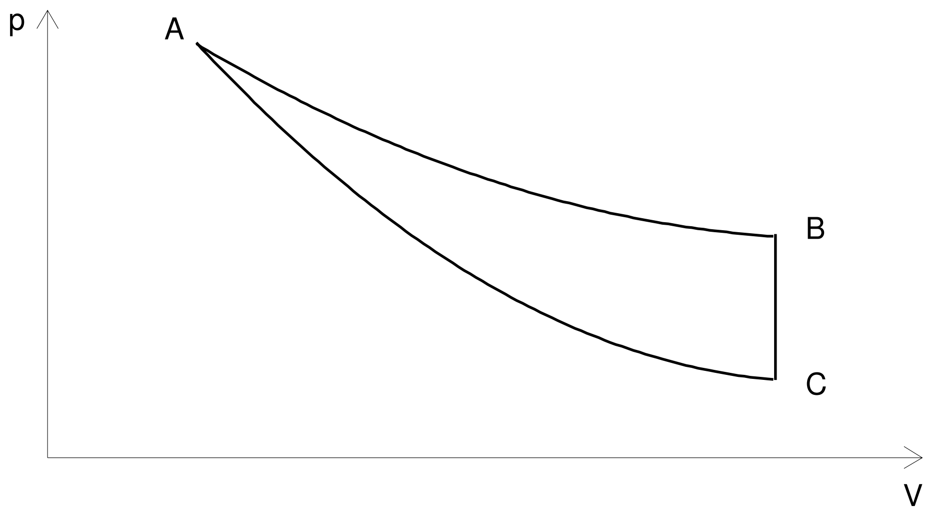
Part IV: Thermal Physics

Do any 2 of the 3 problems.
If you try all 3 problems, indicate clearly which ones you want marked.

Problem 1. A non-ideal gas satisfying the relation

$$E(S,V,N) = \frac{aS^4}{NV^2}$$

where a is a constant is used in the thermodynamic cycle illustrated below, where AB is an isothermal process, BC is an isochore process, and CA is an adiabatic process.



The particle number (N) is constant and the volume ratio $V_C/V_A = 2$.

(a) Show that the equation of state is given by

$$p^3 V = \frac{NT^4}{32\,a}.$$

(b) Calculate the ratio p_B/p_A.

(c) Calculate the ratio p_C/p_A.

(d) Calculate the ratio T_C/T_A.

Problem 2. Consider a classical ideal monoatomic gas of N particles.

(a) What is the work done during an isothermal expansion at temperature T_h from an initial volume V_1 to a final volume V_2?

(b) The gas undergoes now an adiabatic expansion from T_h to T_c. Calculate the work done during this process.

(c) Show that for an adiabatic process $pV^{5/3}$ is constant.
Hint: use the first law of thermodynamics and the ideal gas law.

(d) Calculate the final volume reached after the adiabatic expansion in part (b) in terms of V_2, T_h, and T_c.

Problem 3. A certain amount of water of heat capacity C is at a temperature of 0°C. It is placed in contact with a heat reservoir at 100°C and the two come into thermal equilibrium.

(a) What is the total entropy change of the water plus reservoir system?

(b) The process is now divided into two stages: first the water is brought into contact with a heat reservoir at 50°C and comes into thermal equilibrium; then it is placed in contact with the heat reservoir at 100°C. What is the total entropy change in this case?

(c) If we were to continue this subdivision into an infinite number of heat baths, what would the total entropy change be?
