Qualifying Examination — January 4–5, 2010

General Instructions

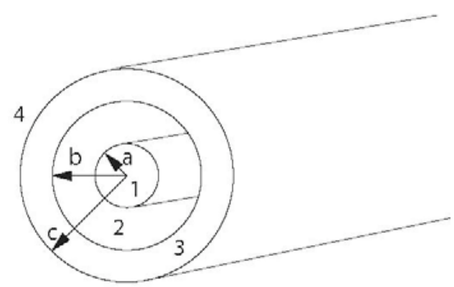
- No reference materials are allowed.
- Do all your work in the corresponding answer booklet.
- On the cover of each answer booklet put only your *assigned number* and the part number/subject.
- Turn in the questions for each part with the answer booklet.
- 90 minutes are allotted for each part, except for Thermal Physics (45 minutes).
- Calculator policy: Use of a graphing or scientific calculator is permitted provided it lacks ALL of the following capabilities: (a) programmable, (b) algebraic operations, and (c) storage of ascii data. Handheld computers, PDAs, and cell phones are specifically prohibited.

Part I: Electricity and Magnetism

Do any 5 of the 6 problems.

1. Write down the real electric field $\vec{E}$ and the magnetic flux density $\vec{B}$ (sometimes called the magnetic induction) as a function of x, y, z, and t for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero, traveling in the direction from the origin (0,0,0) towards the point (1,1,0) with polarization in the z direction. Also, write down the Poynting vector in terms of the same quantities that you used to write down $\vec{E}$ and $\vec{B}$.

2. Assume that an infinite coaxial cable carries opposing currents of equal magnitude I uniformly distributed throughout regions 1 and 3 in the figure below. Take the direction of the current in region 1 to be into the paper. Find $\vec{B}$ (magnitude and direction) in regions 1, 2, 3, and 4.



3. The terminals of a battery having a potential difference of 4.5 V are connected together by a cylindrical metal wire of length 314 m and a diameter of 1 mm. The initial current in the wire is 25 mA.

   (a) Find the resistivity of the wire material.

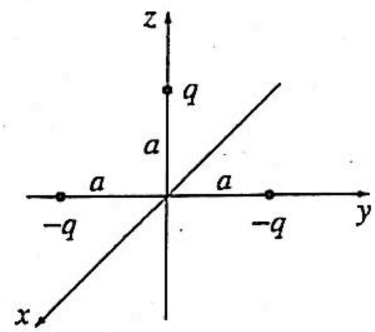
   (b) After a few minutes, the temperature of the wire has increased by 50°C. Find the new resistance of the wire. Given: the temperature coefficient of resistivity for the material is 4.0 $\times 10^{-3}$ per Kelvin.

4. Consider a series LRC circuit. Write down the second-order differential equation for the charge Q on the capacitor. If $R = 0$, find the resonant oscillation frequency ω.

5. Three point charges are located as in the figure below, each at a distance a from the origin.

   (a) Find the monopole and dipole moments.

   (b) Find the corresponding potentials and the electric field at a point far from the origin.



6. (a) Find the $\vec{E}$ and $\vec{B}$ fields, and the charge distributions, corresponding to

$$\Phi(\vec{r},t) = 0 \quad \text{and} \quad \vec{A}(\vec{r},t) = -\frac{qt}{4\pi\epsilon_0 r^2}\hat{r}$$

   (b) Make a gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad \text{and} \quad \Phi' = \Phi - \frac{\partial\lambda}{\partial t}$$

   where

$$\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r}$$

   to transform the scalar and vector potentials and interpret your results.

Part II: Quantum Mechanics

Do any 5 of the 6 problems.

$$\text{Given : } \sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

1. Consider a spin–$\frac{1}{2}$ system for which the Hamiltonian may be represented as

$$H = A \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

where A is a positive constant. The representation of the spin operator in the same basis is

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The state $\chi(t)$ of the system at $t = 0$ is

$$\chi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(a) Are the energy eigenstates simultaneously the eigenstates of any components of the spin operator? Explain.

(b) Find $\chi(t)$.

(c) Calculate the probability that a measurement of the z–component of the spin at time t will yield the result $-\frac{\hbar}{2}$.

2. The potential for a particle of mass m moving in one dimension is

$$V(x) = \begin{cases} \infty & x < 0\,,\ x > a \\ 0 & 0 < x < a/3\,,\ 2a/3 < x < a \\ V_0 & a/3 < x < 2a/3 \end{cases}$$

where V_0 is small. Use perturbation theory to calculate the energy of the first excited state to leading order in V_0.

3. Consider a particle of mass m moving under the influence of the potential $V(\vec{r})$.

   (a) Show that
$$\frac{d}{dt} < O >= \frac{i}{\hbar} < [\hat{H}, \hat{O}] >$$
   where $\hat{H}$ is the Hamiltonian operator, $\hat{O}$ is the operator corresponding to the observable O, and the enclosing angle brackets denote the expectation value. (Assume that $\hat{O}$ has no explicit time dependence.)

   (b) Suppose that the potential is
$$V(\vec{r}) = V_0 \left[\cos(\frac{2\,\pi\,x}{L}) + \cos(\frac{2\,\pi\,y}{L}) + \cos(\frac{2\,\pi\,z}{L})\right]$$
   Using the result of the previous part, obtain the expression for the time rate of change of the expectation value of the momentum for some state $\Psi(\vec{r}, t)$. Show that the result is equal to the expectation value of the force.

4. Find the exact energies of the stationary states for a particle of mass m and charge q moving in a one-dimensional potential
$$V(x) = \frac{1}{2} k\, x^2$$
   which has a uniform electric field of magnitude E_0 pointing in the $+x$ direction superimposed. (Hint: complete the square.)

5. Consider the one–electron eigenfunction
$$\psi_{211} = \frac{1}{8\sqrt{\pi}} \frac{1}{a_0^{5/2}}\, r\, \exp(-r/2\,a_0)\, \sin\theta\, \mathrm{e}^{i\phi}$$

   (a) Show that the eigenfunction is properly normalized.
$$\text{Given}: \quad \int_0^\infty x^n \mathrm{e}^{-ax}\, dx = n!/a^{n+1}$$

   (b) For what value(s) of θ is the probability density maximum? Minimum?

   (c) For what value(s) of ϕ is the probability density maximum? Minimum?

   (d) What is the probability of finding the electron in a thin spherical shell between r and $r + dr$?

6. A small bead of mass m slides without friction on a circular ring of radius R. Let ϕ denote the angular position of the bead. The differential representation of the angular momentum operator is

$$L = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

The bead's wave function at $t = 0$ is

$$\Psi(\phi) = A\,\cos^2\phi$$

For $t = 0$, find the possible outcomes of the measurement of the angular momentum, their probabilities, and the expectation value of the angular momentum. Given:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Part IIIa: Classical Mechanics

Do any 5 of the 6 problems.

1. A block of mass M is dropped onto a spring scale with force constant k. The block is initially at height h above the scale. Answer the following questions in terms of M, g, k, and h. Neglect the mass of the scale.

   (a) How far is the spring compressed when the block reaches its lowest point?

   (b) What is the acceleration of the block when the block reaches its lowest point?

2. Consider a bead of mass m sliding without friction on a wire that is bent in the shape of a cubic curve and is being spun with a constant angular velocity ω about its vertical axis. Use cylindrical polar coordinates and let the equation of the wire be $z = k\rho^3$. Write down the Lagrangian in terms of ρ as the generalized coordinate. Find the equation of motion of the bead and determine whether there are positions of equilibrium.

3. A particle of mass m moves in one dimension under the influence of the potential

$$U(x) = U_0 \left(\frac{a^2}{x^2} - \frac{a}{x} \right).$$

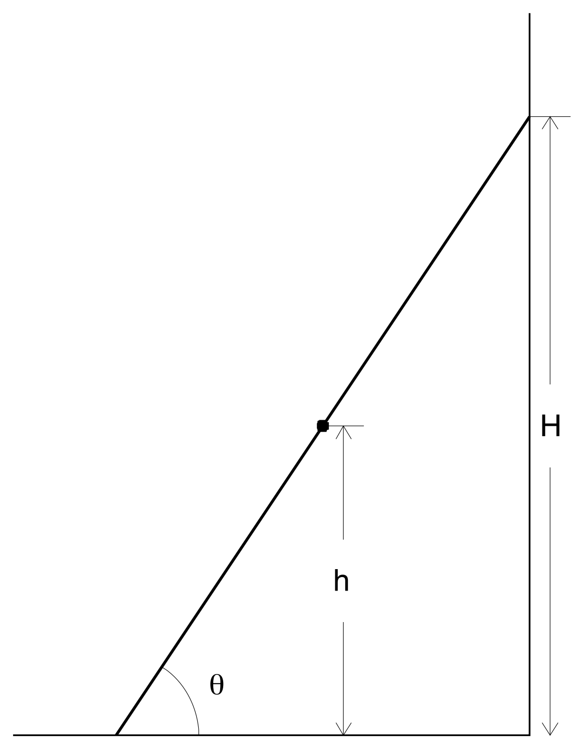
   where U_0 and a are positive constants of appropriate dimensions.

   (a) Sketch this potential in terms of dimensionless variables (i.e., U/U_0 versus x/a) and find the equilibrium position(s).

   (b) Expand about each equilibrium position and calculate the period of oscillations for small displacements.

4. Write down the Lagrangian for the simple pendulum consisting of a massless string of length ℓ and a mass point m. Find the corresponding Euler–Lagrange equation.

5. A person climbs on a lightweight ladder (i.e., of negligible mass) which makes an angle θ with the floor, as illustrated in the figure below. Assuming that the floor has a coefficient of friction μ while the wall has zero coefficient of friction because it is very smooth, calculate the fraction of the maximum height that this person can reach, h/H, without the ladder sliding.



6. Consider a satellite in earth orbit.

   (a) Recall that Kepler's third law states that the orbital period T raised to an integer power is proportional to the orbital semimajor axis a raised to another integer power. Determine the proportionality constant and the exponents by deriving Kepler's third law for the special case of a circular orbit.

   (b) The satellite's orbit has a period of two hours and 4.5 minutes. At perigee ($r = r_{\text{min}}$) it is observed to be 250 km above the Earth's surface, traveling at about 8500 m/s. Using Kepler's third law, calculate the height above Earth at apogee ($r = r_{\text{max}}$) , as well as the speed at this point.

   Given: The Earth's radius is $R_e = 6.4 \times 10^6$ m. Also note that $GM_e/R_e^2 = g$ and that

$$a = \frac{1}{2}\left(r_{min} + r_{max}\right).$$

Part IIIb: Thermal Physics

Do any 2 of the 3 problems.

Given : R = 8.31 J/(mol K) $k_B = 1.38 \times 10^{-23}$ J/K $N_A = 6.02 \times 10^{23}$ 1 L = 10^{-3} m^3

1. 50 g (2 mol) of an ideal gas have a volume V_i at pressure P_i and a specific heat c. This gas is running through the following 3–step process.

   i) At constant volume 200 J of heat are added to the gas.

   ii) Next, at constant temperature, the gas is expanded until it reaches its original pressure P_i.

   iii) Finally, the gas is compressed at constant pressure until it reaches its original volume V_i.

   The numerical values are

   $$V_i = 10 \text{ L} \quad P_i = 10^5 \text{ Pa} \quad c = 500 \text{ J/(kg K)}$$

   (a) Sketch the P vs V diagram for the steps of this process.

   (b) Calculate the temperature of the gas after completion of the first step.

   (c) Calculate the volume of the gas after completion of the second step.

   (d) Calculate the amount of work done on the gas in the third step.

   (e) During the third step: does the temperature rise, drop or stay constant? Explain.

   (f) For the entire 3-step process: does the gas do work on the exterior or is the exterior doing work on the gas? Explain.

2. A container of volume 2 L has a center partition that divides it into two equal parts. The left side contains hydrogen gas and the right side contains oxygen gas. Both gases are at a temperature of 300 K and a pressure of 10^5 Pa. The partition is removed to allow the gases to mix. Calculate the change in entropy of the system. The initial and final temperatures of the system are the same.

3. Consider a gas in thermal equilibrium comprised of N identical spin–$\frac{1}{2}$ point particles of mass m confined to a cube of side L. (Such particles obey Fermi–Dirac statistics.) Assume that N is on the order of Avogadro's number and L is on the order of 10 cm. Neglecting interactions, the allowed one–particle energies are given by

$$E = \frac{\pi^2 \hbar^2}{2mL^2}(n_1^2 + n_2^2 + n_3^2) \quad n_1 = 1, 2, 3, ... \quad n_2 = 1, 2, 3, ... \quad n_3 = 1, 2, 3, ...$$

   (a) Find the number of one–particle states with energy between E and $E + dE$. Express your answer in terms of E, m, V $(= L^3)$, and physical and mathematical constants.

   (b) Calculate the mean energy of the particles at absolute zero.

   (c) Explain why the molar heat capacity of this system at room temperature is much smaller than $\frac{3}{2}R$.