QUALIFYING EXAMINATION – JANUARY 2005

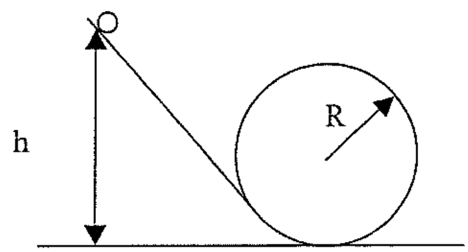
General Instructions

No reference materials are allowed, except for the use of a calculator. Do all your work in the answer booklet. On the cover of each answer booklet, put only your *assigned number* and the part number. Turn in the questions for each part with the answer booklet. 90 minutes are allotted for each part, with a 30 minute break in between..

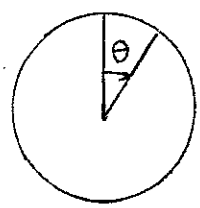
Part I. Classical Mechanics

Do any 5 of the 6 problems.

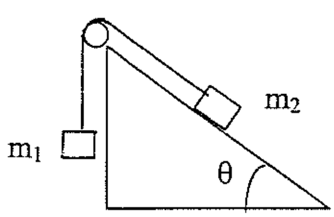
1. A sphere of mass m, radius r, and moment of inertia $I = kmr^2$ (where k is a dimensionless constant) is released from a height h on a loop-the-loop track as shown. The radius of the loop is R. What is the minimum value of h for which the ball makes it all the way around the loop? You may assume r<<R, but indicate at what point(s) in your analysis you are making that assumption.



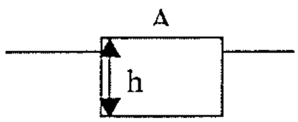
2. A particle is placed at the top of a smooth (frictionless) sphere of radius R. If the particle is slightly displaced, at what angle θ from the vertical will it leave the surface of the sphere?



3. Two blocks are connected by an ideal massless string that passes over a massless, frictionless pulley. The second mass lies on an incline of angle θ. The coefficient of static friction between the block and the incline is μ_s. For what values of m_1 (give both a minimum and a maximum) is this system in equilibrium (at rest)?

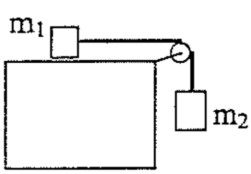


4. A rectangular block of height h and area A floats in water. If the density of the block is ρ_b and the density of the water is ρ_w, find the frequency of small oscillations when the block bobs up and down on the surface of the water.



5. A particle moves in a circular orbit under the influence of a central force, $f(r) = -cr^n$. Find the condition on n for which this orbit is stable.

6. Masses m_1 and m_2 are connected by a uniform string of fixed length L and mass M. The string passes over a massless pulley as shown. There is no friction.



(a) Write down the Lagrangian for this system.

(b) Find the equation of motion.

(c) Solve the equation of motion to find the position of m_2 as a function of time, assuming it starts from rest at the pulley.

Qualifying Examination – 2005
Part II. Electricity and Magnetism
Do any 5 of the 6 problems. (See last page for useful equations.)

1) A wire in the shape of an equilateral triangle of length L on each side lies in the xy plane and carries current I. What is the $\vec{B}$ field at large distances from the wire?

2) In the region of positive z, a scalar electric potential has the form

$$\Phi = \Phi_0 \frac{\sin(kr)}{kr} e^{-kz}$$

where r is the distance from the z axis. What is the charge density in the region?

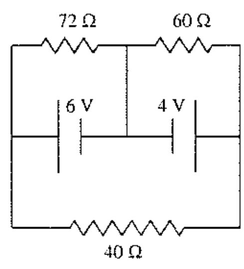
3) A square wire loop of length L on each side lies in the xy plane with one corner at the origin and sides parallel to the x or y axes. In the region of the loop there is a time varying $\vec{B}$ field of the form

$$\vec{B} = \vec{e}_z B_0 \frac{x^2 + y^2}{L^2} e^{-t/\tau}$$

What is the induced $\mathcal{EMF}$ in the loop?

4) A current density $\vec{J} = J_0 \frac{a}{s} e^{-s/a} \vec{e}_z$ (where s is the distance from the z axis) fills all of space. What is the $\vec{B}$ field everywhere in space?

5) What are the currents (magnitudes and directions) through the two batteries in the circuit shown?



6) Four charges of strength q are fixed at positions $(\pm a, 0)$ and $(0, \pm a)$ in the x, y plane. A fifth charge of strength q is at the origin. If this fifth charge is displaced a small amount in the x direction, with what angular frequency will it oscillate?

Useful formulae

In spherical, cylindrical, and cartesian coordinates respectively, the Laplacian is

$$
\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\
\nabla^2 &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \\
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

Qualifying Examination – 2005
Part III. Quantum Mechanics

Do any 5 of the 6 problems. (See last two pages for useful equations and quantities.)

1. A particle is known to be in the state

$$\psi(x,t) = A\exp\left[-\frac{(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0x}{\hbar}\right)\exp(i\omega t)$$

where x_0, a, p_0 and ω are constants.

a) Find $< x >$

b) Find $< p >$

2. A particle of mass m is in the one-dimensional potential

$$V = \begin{cases} \infty, & x \le 0 \\ 0, & 0 < x < a \\ V_0, & x > a\,. \end{cases}$$

a) Show that in the region $0 < x < a$ the wave function is of the form

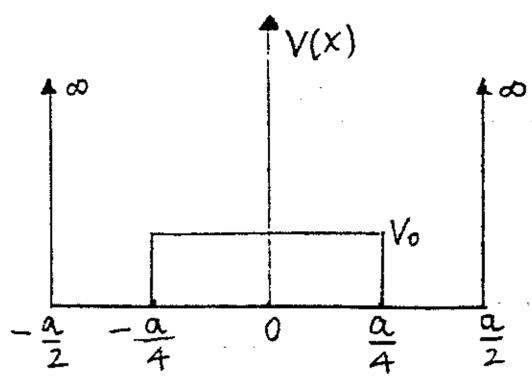
$$\psi(x) = A\sin(kx) + B\cos(kx)$$

where $k = \frac{\sqrt{2mE}}{\hbar}$

b) Find the wave function in the region $x > a$ for $E < V_0$.

3. A small perturbing potential is added to the infinite square well potential as shown below. Use perturbation theory to calculate the lowest energy eigenvalue E_1 for the potential (assume that $V_0 << E_1^{(0)}$ where $E_1^{(0)}$ is the unperturbed lowest energy eigenvalue)

$$V(x) = \begin{cases} \infty & x < -a/2,\ x > a/2 \\ 0, & -a/2 < x < -a/4,\ \ a/4 < x < a/2 \\ V_0, & -a/4 < x < a/4\,. \end{cases}$$



4. The function

$$\psi(r,\theta,\phi) = \frac{1}{8\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\frac{Zr}{a_0}e^{-Zr/2a_0}\sin(\theta)e^{i\phi}$$

is the wave function for a particular quantum system.

a) Is this an eigenfunction of the orbital angular momentum operator L_z? If so, what is the eigenvalue for this operator? If not, what modification to ψ would be required to make ψ an eigenfunction?

b) Same questions as a) for the square of the total orbital angular momentum operator L^2.

5. a) Calculate the difference in energy between the two allowed electron-spin orientations in a uniform magnetic field $\vec{B}$?

b) What is the frequency of radiation that can induce transitions (spin flips) between these two states when $B = 0.5T$?

6. A harmonic oscillator is in the eigenstate given by

$$\psi_n = A(8\eta^3 - 12\eta)e^{-\eta^2/2}$$

where $\eta = \sqrt{\frac{m\omega_0}{\hbar}}x$, $A = (48\sqrt{\pi})^{-1/2}$.

a) What is the energy of the oscillator in this state?

b) What is the value of the quantum number n?

Useful Relations and Constants

Laplacian in spherical coordinates

$$\frac{\partial^2 f(r,\theta,\phi)}{\partial r^2} + \frac{2}{r}\frac{\partial f(r,\theta,\phi)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f(r,\theta,\phi)}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f(r,\theta,\phi)}{\partial\phi^2}$$

Laplacian in cylindrical coordinates

$$\frac{\partial^2 f(\rho,\phi,z)}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f(\rho,\phi,z)}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} f(\rho,\phi,z) + \frac{\partial^2 f(\rho,\phi,z)}{\partial z^2}$$

Useful Integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha y^2} dy = \sqrt{\pi/\alpha}$$

Constants

Planck's constant $= 6.63 \times 10^{-34}$ J$\cdot$ s
Charge on the electron $= 1.6 \times 10^{-19}$ C
Mass of the electron $= 9.1 \times 10^{-31}$ Kg
Bohr magneton $= \frac{e\hbar}{m} = 0.927 \times 10^{-23}$ J/Tesla

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

$$\psi_{200} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(2 - \frac{Zr}{a_0}\right) e^{-Zr/2a_0}$$

$$\psi_{210} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \cos\theta$$

$$\psi_{21\pm1} = \frac{1}{8\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \sin\theta\, e^{\pm i\phi}$$

$$\psi_{300} = \frac{1}{81\sqrt{3\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(27 - 18\frac{Zr}{a_0} + 2\frac{Z^2r^2}{a_0^2}\right) e^{-Zr/3a_0}$$

$$\psi_{310} = \frac{\sqrt{2}}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(6 - \frac{Zr}{a_0}\right)\frac{Zr}{a_0} e^{-Zr/3a_0} \cos\theta$$

$$\psi_{31\pm1} = \frac{1}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(6 - \frac{Zr}{a_0}\right)\frac{Zr}{a_0} e^{-Zr/3a_0} \sin\theta\, e^{\pm i\phi}$$

$$\psi_{320} = \frac{1}{81\sqrt{6\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0}(3\cos^2\theta - 1)$$

$$\psi_{32\pm1} = \frac{1}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0} \sin\theta\cos\theta\, e^{\pm i\phi}$$

$$\psi_{32\pm2} = \frac{1}{162\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0} \sin^2\theta\, e^{\pm 2i\phi}$$

Qualifying Examination – 2005
Part IV. Mixed Topics

Do problems from 4 of the 5 subsections (choose from either the astrophysics subsection or the electronics subsection). Do 6 of the 8 problems of these subsections, with at least one problem in each of the 4 subsections.

A. Special Relativity

1. A particle has a speed $c/2$ in frame S and is moving in the positive x direction. Another frame, S', is moving along $+x$ at speed $3c/4$. What is the velocity of this particle in S'?

2. μ meson particles are produced by cosmic rays 20km up in the Earth's atmosphere. Take the μ mesons to be traveling straight downward at $v/c = 1 - \epsilon$, where ϵ is a small number. Of the number of mesons produced in the upper atmosphere, approximately 1/8 survive to reach the surface of the Earth. In between, the ones that do not reach the surface decay with a meson half-life of 2.2×10^{-6} sec. Find the value of ϵ.

B. Thermal Physics

1. Suppose a heat engine is connected to energy reservoirs, one a pool of molten aluminum at its freezing temperature (660 ^oC), and the other a block of solid mercury at its melting temperature (-39 ^oC). The engine runs by freezing 1 g of aluminum and melting 15 g of mercury during each cycle. The latent heat of fusion of aluminum is 3.97 x 10^5 J/kg, and that of mercury is 1.18 x 10^4 J/kg.

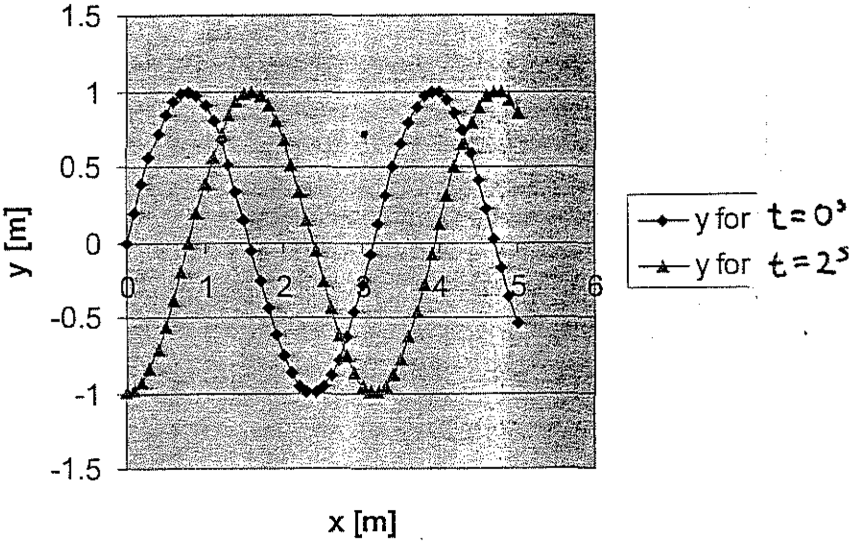
a) Find the efficiency of this engine.
b) Calculate the efficiency of a Carnot engine operating between the same temperatures, and compare with the result of (a).

2. Consider a system of cobalt nanoparticles each containing 1600 cobalt atoms whose magnetic moments are fully aligned at zero temperature. Hence each nanoparticle can be thought of as a small ferromagnet with a moment **M** whose magnitude M equals the total moment of the cobalt atoms. (Each cobalt atom has a moment of 1.6 μ_B, where μ_B is a Bohr magneton = 0.9274 x 10^{-23} J/Tesla.) Assume that the variation of **M** with temperature can be ignored. The nanoparticles are placed in a uniform external magnetic field **H** which points in the z-direction, so that the interaction energy of a single nanoparticle is $-\mathbf{M} \cdot \mathbf{H}$. Ignore the interaction between the nanoparticles.

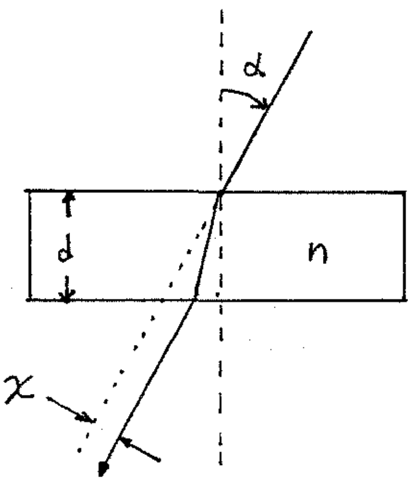
Find the average magnetization (i.e., the z-component of the magnetic moment) per nanoparticle that would be observed in room temperature (300 K) and in a field of 1 Tesla. (Boltzmann constant k_B = 1.38 x 10^{-23} J/K).

C. Optics

1. A wave is travelling to the right. The plot shows two snapshots, one taken at t=0, the other taken at t=2s. Find the values of the amplitude, wave velocity, wave vector k, and the period T. (Read necessary values from the plot with reasonable accuracy.)



2. A light beam enters a slab of glass (index of refraction n; thickness d=10mm) under an angle of α=30°. The exit beam is displaced sideways parallel to the direction of the original beam by x=2mm. Calculate the index of refraction of the glass.



D. Astrophysics

Possibly useful information. The energy output per second of the Sun is 3.90×10^{26} W. Constants σ=5.67×10^{-8} W m^{-2} K^{-4}; k=1.38×10^{-23} J/K; c=3×10^{8} m s^{-1}. 1 $M_\odot$=1.99×10^{30} kg.

1. The most massive main sequence stars are very luminous and blue in color. For example, B0 V stars have absolute bolometric magnitudes of -7.2 and masses of $18M_\odot$. The absolute bolometric magnitude of the Sun is $+4.7$.

(a) What is the luminosity of a B0V star? Specify in units of the Sun's luminosity.
(b) With simple assumptions and the information above, estimate the main sequence lifetime of this star compared to the Sun.

2. Our Sun is a stable star having endured at about the same size, surface temperature, and luminosity for over 4 billion years.

(a) In terms of the density $\rho(r)$, mass interior to radius r, $M(r)$, and the gravitational constant G, write an expression for the inward gravitational force on a shell of thickness dr in the Sun.
(b) Define $P(r)$ as the pressure as a function of radius r. In terms of this variable, what is the net outward pressure force on the above shell?
(c) From our knowledge about the Sun, what must be the equation relating the two expressions you obtained in parts (a) and (b) in the Sun?

E. Electronics

1. When measuring voltages in circuits which have high value resistors, one must take care to account for the internal resistance of the meter to properly interpret the results. To illustrate this, consider the following circuit. Two identical 5.00 MΩ resistors are connected in series with a 15.0 V battery. A voltmeter with an internal resistance of 10.0 MΩ is used to measure the voltage drop across one resistor. In this particular case, a significant reduction occurs in the resistance that voltage is measured across.

a. If the meter has no effect other than to measure voltage, what voltage would you expect to measure across one of the 5.00 MΩ resistors?
b. Calculate the resistance of the parallel combination of the 10.0 MΩ meter and the 5.00 MΩ resistor.
c. Calculate the voltage across the parallel combination of meter and resistor.
d. What is the maximum value of resistance that the 10 MΩ meter could be used to measure a voltage drop, in order that the voltage be within 1% of the value without the meter?

2. For the circuit below, assume L=20.0mH, R=470Ω, and C=0.010μF.

a. For an AC current i=(25.0mA)cos[$2\pi(1.5kHz)t$], find the voltage across the resistor.
b. What is the resonance frequency in Hz of the circuit?
c. What should the frequency be to make the voltage across the resistor 1/10 the voltage across the capacitor?