QUALIFYING EXAMINATION
JANUARY 2000

General Instructions: No reference materials (except for the use of a calculator) are permitted. Do all your work in the answer booklet. Turn in the questions for each part with the answer booklet. There are 90 minutes allotted for each part, with a 30-minute break in between.

PART I: Classical Mechanics

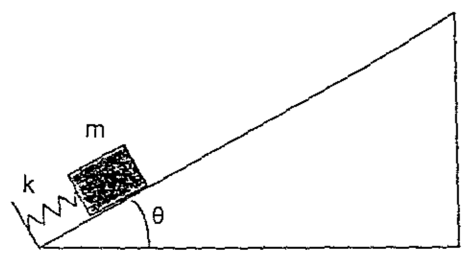
Work any 5 of the 6 problems.

1. Three gliders of mass m, 2m, and 4m sit at rest on a frictionless air track at positions 0, 50 cm, and 100 cm, respectively. The first glider, of mass m, is given an initial speed of 20 cm/s toward the next glider. What is the speed of the third glider if all collisions are elastic?

2. Halley's comet moves about the sun in an elliptical orbit with perihelion 8.9×10^{10} m and aphelion 5.3×10^{12} m. What is the speed of the comet at perihelion (closest to the sun) and at aphelion (farthest)? ($G = 6.67 \times 10^{-11}$ N-m^2/kg^2, $M_{sun} = 1.99 \times 10^{30}$ kg)

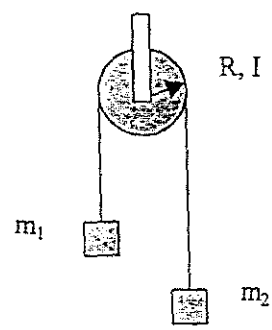
3. A particle of mass m = 2 kg has a potential energy given by $V(x) = 30 - 10x^3 + 4x^4$, where x is in meters and V is in joules. Locate the equilibrium position(s) and find the frequency for small oscillations about the equilibrium position(s).

4. A block rests against a spring on an inclined plane, as shown to the right. The spring has force constant k = 100 N/m, the mass of the block is 0.2 kg, and the coefficient of friction between the block and the incline is 0.3. The block is pushed against the spring, compressing it by an amount 0.15 m, and released from rest. How far up the incline does the block travel?



5. Derive an expression for the rotational inertia of a uniform rectangular sheet of mass M with sides a and b about an axis that is perpendicular to the sheet and through one of its corners.

6. An Atwood machine consists of two masses ($m_1 > m_2$) connected by a cord of length L that hangs over a pulley. The pulley has radius R and rotational inertia I. Assume that the cord is massless and there is no slippage. Use Lagrange's equations to solve for the acceleration of the masses.



PART II. **Electricity and Magnetism**
(work five problems)

Vectors are denoted by bold print.

1. Find the electrostatic potential inside and outside a uniformly charged sphere of radius R with total charge Q. Set the potential at infinity equal to zero and give your answer in terms of Q and R.

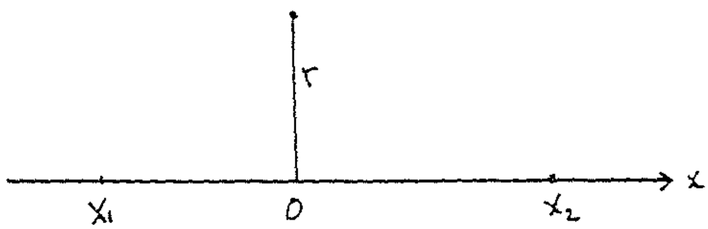
2. The potential on the surface of a sphere of radius R is given by $V(R,\theta) = V_o\cos(\theta)$ where V_o is a constant and θ is the polar angle. Find the potential everywhere assuming that the only charge is on the surface of the sphere. Hint: A solution to Laplace's equation with azimuthal symmetry has the form

$$V(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})P_n(\cos\theta)$$

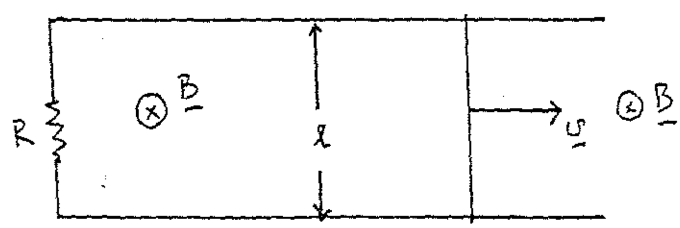
where $P_n(x)$ are the Legendre polynomials. $P_0(x) = 1$, $P_1(x)=x$, $P_2(x)=(3x^2-1)/2$, etc.

3. A sphere of radius R has a polarization vector **P** = k**r** where k is a constant. (a) Assuming no free charges, calculate the surface and volume charge densities due to the polarization. (b) Find the electric field and the electric displacement vector inside and outside the sphere. (c) Verify that the total charge is zero.

4. Use the Biot-Savart law to find the magnetic field a distance r away from a straight conducting wire segment lying along the x-axis with ends at x_1 and x_2 carrying a current I using the diagram below. Find the field in the limit of a wire of infinite length.



5. A superconducting bar slides without friction on two superconducting rails a distance l apart. A resistance R is connected across the rails and a uniform magnetic field **B** pointing into the page fills the whole region. (a) If the bar moves to the right with velocity v, use Faraday's law to determine the current (magnitude and direction). (b) What force is exerted on the bar due to the magnetic field? (c) If the bar has velocity v_0 at t=0, find its velocity as a function of time. (d) Show that energy is conserved.

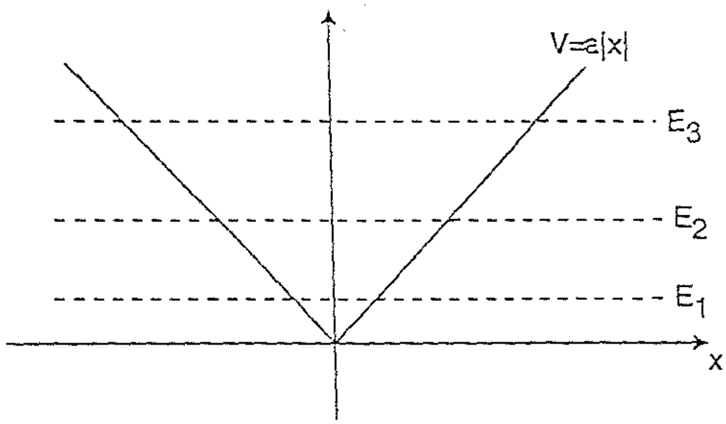


PART III. Quantum Mechanics

Work 5 problems

1. Consider a large number of one–particle systems each moving under the influence of the same potential. Each of these systems has been prepared so that it is described by the same wave function $\Psi(x,t)$. The energy eigenfunctions are $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$,.... with respective energies E_1, E_2, E_3,..... At some time t, the energy of each of these systems is measured. For one half of the measurements, the energy E_1 is obtained, in one quarter of the cases the energy E_2 is obtained, and the energy E_5 is obtained by the remaining measurements. In terms of $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$..... and E_1, E_2, E_3,....., write down an expression for $\Psi(x,t)$ which is consistent with these measurement results. Explain how you arrived at this expression.

2. A particle moves in one dimension under the potential sketched below. Also indicated in the sketch are the energies of the ground state, first excited state, and second excited state. Sketch the eigenfunctions for each of these states. Discuss how the properties of this one-particle system differ from what is expected from classical mechanics. Also comment on the connections, if any, between the symmetries of the system and the properties of the eigenfunctions.



3. For some particular choice of basis, the Hamiltonian for a system is represented by the matrix

$$E_0 \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$$

where E_0 is real and positive. (a) Find the possible outcomes of an energy measurement on this system. (b) Suppose that the state of the system at t=0 is

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Find the expectation value for the energy and the probability that a measurement of the energy will yield the ground state energy.

$$\int \cos x dx = \sin x$$

$$\int \sin x dx = -\cos x$$

$$\int x \sin^2 x dx = \frac{1}{4} \sin x(\sin x - 2x \cos x) + \frac{1}{4}x^2$$

$$\int x \cos^2 x dx = \frac{1}{4} \cos x(\cos x + 2x \sin x) + \frac{1}{4}x^2$$

$$\int x^2 \cos^2 x dx = \frac{1}{2}x \cos^2 x + \frac{1}{2}x^2 \sin x \cos x + \frac{1}{6}x^3 - \frac{1}{4}x - \frac{1}{4} \cos x \sin x$$

$$\int x^2 \sin^2 x dx = \frac{1}{2}x \sin^2 x - \frac{1}{2}x^2 \sin x \cos x + \frac{1}{6}x^3 - \frac{1}{4}x + \frac{1}{4} \cos x \sin x$$

$$\int \cos^2 x dx = \frac{1}{2}(x + \sin x \cos x)$$

$$\int \sin^2 x dx = \frac{1}{2}(x - \sin x \cos x)$$

$$\int \sin x \cos x dx = -\frac{1}{4} \cos 2x$$

PART IV--MIXED TOPICS

Do problems from 4 of the 5 sections (choose either the astrophysics or electronics section). Do 6 of the 8 problems in these sections, with at least one from each of the 4 sections. Use a different answer book for each lettered section.

A. Relativity

Represent the speed of light (in special relativity) with the symbol, $c = 3 \cdot 10^8$ m/s.

1. μ meson particles are produced by cosmic rays 60 km up in the earth's atmosphere. Take the mesons to be traveling straight downward at close to the speed of light i.e. $v/c = 1 - \varepsilon$ where ε is a small number.

   Of the number of mesons produced in the upper atmosphere approximately 1/8 survive to reach the surface of the earth. In between, the ones that do not reach the surface decay to other particles with a meson half-life of $1.5 \cdot 10^{-6}$ sec. How many km/s less than the speed of light are the mesons traveling as measured by an observer on the earth, i.e. what is ε?

2. An observer in a vacuum travels at a speed v m/s past a light source located at 90 degrees to the observer's direction of motion. The light source is a distance x away from the observer at closest approach. Derive an expression for the angle at which the light source is apparently shifted away from 90 degrees at closest approach due to the motion of the observer.

C. Optics

1. The index of refraction of water is 1.333, the index of refraction of benzene is 1.50, and the speed of light in vacuum is 3×10^8 m/s. Monochromatic light has a wavelength of 500 nm in vacuum.

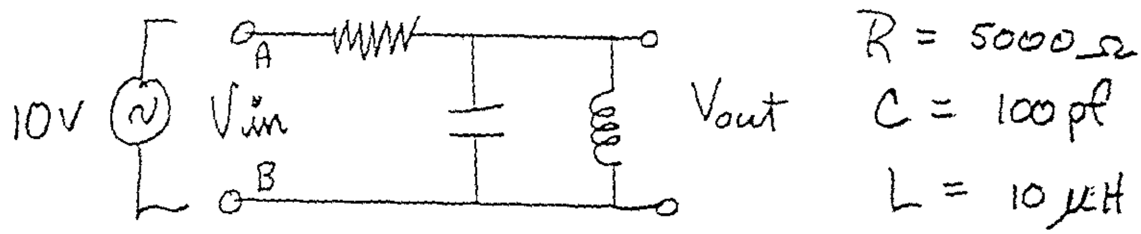
   a. What is the frequency of this light in vacuum?
   b. What is the frequency of this light in benzene?
   c. What is the speed of this light in water?
   d. What is the wavelength of this light in water?
   e. If light travels from benzene into water at an angle of incidence of $30°$ what is the angle of refraction?
   f. At what angle of incidence will total internal reflection occur if light is incident from benzene to water?

2. Two radio towers are 8 meters apart. They emit radio waves of wavelength 1 meter; the waves emitted by the two towers are exactly in phase.

   a. If you are 100 meters from a point directly between the two towers and on the perpendicular bisector connecting them, what kind of interference will you observe?
   b. If you are 100 meters from the point between the towers and on the line connecting them, what kind of interference will you observe?
   c. If you start at the point in part a and walk slowly along the arc of a 100 meter (radius) circle to the point in part b, how many maxima and minima will you observe?

E. Electronics

1. Consider the circuit shown below:

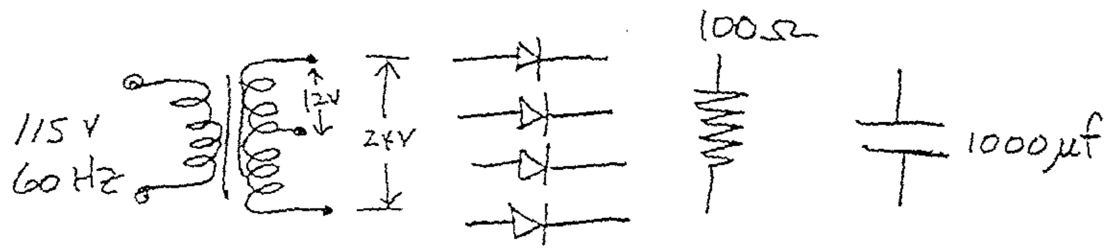


a. Derive an expression for the impedance between A and B (use complex numbers).

b. For what input frequency will the output voltage be largest?

c. Sketch a graph of the expected output voltage as a function of input frequency.

2. Consider the circuit elements shown below:



a. Draw a circuit using these components which will give a full-wave rectified output with a resistor as the load.

b. Draw the expected waveform across the output resistor.

c. Add the capacitor to the circuit in such a way as to provide filtering of the waveform, and draw the expected waveform across the resistor.