QUALIFYING EXAMINATION
FALL 1993

General instructions. No reference materials (other than a calculator) are permitted. Do all work in your answer booklet. Turn in the questions for each part with the answer booklet. You may finish Part I early, turn it in and start work on Part II.

PART I -- CLASSICAL MECHANICS

Work any 5 of the 6 problems.

1. Use Lagrange's equations to find the differential equation of motion of a mass M attached to the lower end of a vertical spring with spring constant k. (The mass is constrained to move in a vertical straight line.) Find the frequency for oscillating solutions.

2. The potential energy of a particle is $V = ax^2 - bx^3$. The particle starts from the origin along the +x axis with an initial velocity v_0. What is the maximum value of v_0 for which the particle remains trapped (i.e., confined to a region near the origin.)?

3. A chain of total length L and total mass M is partially hanging over the corner of a frictionless table. If it is released from rest with the segment of length ℓ hanging, what is the speed of the chain when the last link clears the top? (Assume the table is higher than L.)

4. A small ball swings in a horizontal circle at the end of a cord of length L_1 which forms an angle θ_1 with the vertical. The cord is slowly shortened by pulling it through a hole in its support until the free end has length L_2 and the ball is moving such that the cord makes an angle θ_2 with the vertical. Show that

$$(L_1 \sin \theta_1)^3 \tan \theta_1 = (L_2 \sin \theta_2)^3 \tan \theta_2$$

5. A solid cylinder of mass M and radius R has a few turns of light string wound around it. If the end of the string is held steady and the cylinder is allowed to fall under gravity, what is the acceleration of the center of mass of the cylinder?

6. A cylinder of mass M and radius R is initially projected so as to slide and roll along a rough horizontal plane with coefficient of kinetic friction μ. If the initial linear velocity of the center of mass of the ball is v_0 and the initial angular velocity about the center of mass is ω_0, what is the linear speed of the center of mass when pure rolling begins? Assume that $R\omega_0 < v_0$.

PART II -- ELECTRICITY AND MAGNETISM

Work any 5 of the 6 problems. You may use either SI or Gaussian (cgs) units.

SI Constants

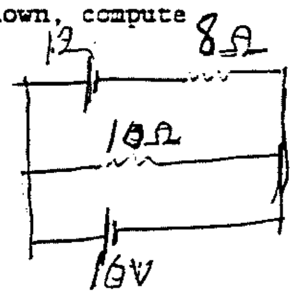
$\epsilon_0 = 8.85\times10^{-12}\ C^2/Nm^2$ $\mu_0 = 4\pi\times10^{-7}\ N/A^2$ $m_e = 9.1\times10^{-31}$ kg

1. A parallel-plate capacitor with a gap of 1 cm has a 0.5 cm slab of material with $\epsilon = 3\epsilon_0$ against one plate. The rest of the gap is filled with air ($\epsilon = \epsilon_0$).

(a) Compute the electric field in each half of the capacitor if the potential difference between the plates is 1000 volts.

(b) Compute the capacitance of the capacitor if its area is 0.2 m^2.

2. In the circuit shown, compute



(a) the current through the 10Ω resistor,

(b) the current through the 10V cell.

3. An air-filled solenoid has radius 1.0 cm, length 10 cm, and has 1000 turns of wire carrying 2 amperes.

(a) Compute the magnetic field in the solenoid.

(b) A cylinder of length 10 cm, radius slightly smaller than 1 cm, and magnetic permeability $\mu = 1.5\ \mu_0$ is inserted half way (5 cm) into the solenoid. Compute the force on the cylinder.

4. (a) Derive from Maxwell's equations the relation between the electric field amplitude and the magnetic field amplitude in a plane electromagnetic wave in a vacuum.

(b) Compute E_0 if B_0 = 1T.

5. An electron has velocity 3×10^6 m/s perpendicular to a magnetic field, and is observed to move in a circle of radius 0.3 m.

(a) What is the strength of the B field?

(b) What E field could you apply (in addition to the B field) to cause the electron to move in a straight line instead? Give magnitude and direction.

6. An infinitely thin line charge (charge/length = λ) is parallel to a grounded conducting cylinder of radius R. The distance between the line charge and the axis of the cylinder is b (b>R). Compute the force per unit length on the line charge. Check that your result reduces to the correct limit as $b/R \to \infty$.

PART III -- QUANTUM MECHANICS

Answer any 5 questions.

1. A particle is known to be in the state with the wave function given by

$$\psi(x,t) = A \exp[-(x-x_o)^2/(4a^2)] \exp(ip_o x/\hbar)\exp(i\omega_o t)$$

where x_o, a, p_o and ω_o are constants.

(a) Calculate A.

(b) What is the probability density for this state?

(c) Calculate the mean-square deviation of the position $(\Delta x)^2$.

Useful Integral $$\int_{-\infty}^{\infty} \exp(-y^2)dy = 2\pi$$

2. Consider the functions

$$\phi_K = (1/\sqrt{L})\, e^{ikx}$$

definded over the interval $(-L/_2, L/_2)$.

(a) Show that these functions are normalized to unity and maintain this normalization in the limit $L \to \infty$.

(b) Show that the functions comprise an orthogonal set in the limit $L \to \infty$.

3. A harmonic oscillator is in the eigenstate given by

$$\psi_n(y) = A(8y^3-12y)\exp(-y^2/2)$$

where $y = \sqrt{\frac{m\omega_o}{\hbar}}\, x$, $A = (48\sqrt{\pi})^{-1/2}$.

(a) What is the energy of the oscillator in this state?

(b) How many energy eigenstates are below this energy?

4. A D_2 molecule at 30K, at t=0, is in the state

$$\psi(\theta,\phi) = \frac{3Y_1^1 + 4Y_7^3 + Y_7^1}{26}$$

where Y_ℓ^m is the spherical harmonic function.

(a) What possible values of the total angular momentum operater squared (L^2) will measurement find and with what probabilities will these values occur?

(b) What is $< L^2 >$ for this state?

(c) Same questions as in (a) for the operator L_z.

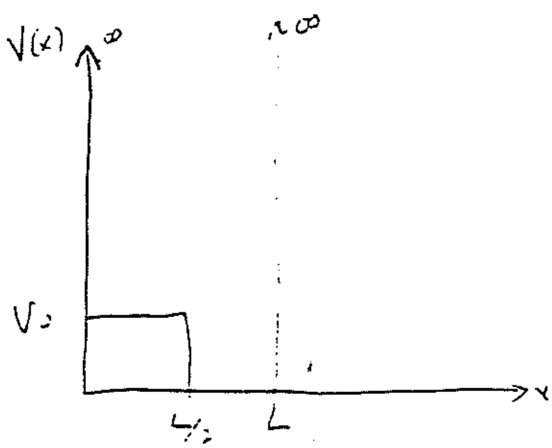
(d) What is $< L_z >$ for this state?

5. A sample of hydrogen atoms all in the first excited energy level with energy E_2 is measured to determine the probability density. Assuming that all of the degenerate states of this level are equally probable, determine the probability density function. A table of hydrogen atom wave functions is given at the end of this section.

6. A particle of mass m is in an asymmetrical one-dimensional box, shown below.

(a) Use first-order perturbation theory to calculate the eigenenergies of the particle.

(b) What are the first-order corrected wave functions?



$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

$$\psi_{200} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(2 - \frac{Zr}{a_0}\right) e^{-Zr/2a_0}$$

$$\psi_{210} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \cos\theta$$

$$\psi_{21\pm1} = \frac{1}{8\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \sin\theta\, e^{\pm i\phi}$$

$$\psi_{300} = \frac{1}{81\sqrt{3\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(27 - 18\frac{Zr}{a_0} + 2\frac{Z^2r^2}{a_0^2}\right) e^{-Zr/3a_0}$$

$$\psi_{310} = \frac{\sqrt{2}}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(6 - \frac{Zr}{a_0}\right)\frac{Zr}{a_0} e^{-Zr/3a_0} \cos\theta$$

$$\psi_{31\pm1} = \frac{1}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(6 - \frac{Zr}{a_0}\right)\frac{Zr}{a_0} e^{-Zr/3a_0} \sin\theta\, e^{\pm i\phi}$$

$$\psi_{320} = \frac{1}{81\sqrt{6\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0}(3\cos^2\theta - 1)$$

$$\psi_{32\pm1} = \frac{1}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0} \sin\theta\cos\theta\, e^{\pm i\phi}$$

$$\psi_{32\pm2} = \frac{1}{162\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0} \sin^2\theta\, e^{\pm 2i\phi}$$

PART IV -- MIXED TOPICS

Do problems from 4 of the 5 sections. Astrophysics may be chosen only in place of electronics. Do 6 of the 8 problems with at least one from each of the 4 chosen sections.

A: Relativity

1. What is the speed of a massive relativistic particle which has kinetic energy equal to its rest energy?

2. Frame F′ is moving with velocity v in the x-direction of frame F. An object is moving with velocity $u = (u_x', u_y', u_z')$ relative to frame F′. What is the y-component of its velocity u_y in frame F?

B: Thermal Physics

1. What is the typical speed of a helium atom in the atmosphere, and how far out on the tail of the velocity distribution function does a helium atom need to be for escape? Consider the major constituents of the atmosphere to be heavy molecules with molecular weight 29 AMU (weighted mean of diatomic nitrogen and oxygen), with helium only as a trace. The heavy component has RMS velocity at mean atmospheric temperature of 500 m/s (take the atmosphere to be isothermal). Take the Earth's radius to be 6000 km; note that escape velocity at small heights above the surface is 10.8 km/s.

2. (a) Heat cannot be converted to work entirely. Why?

(b) Calculate the efficiency of an ideal engine operating between the temperatures characteristic of an automotive engine, say 300° C and room temperature.

C: Optics

1. A concave mirror has a radius of 40 cm. Find the two distances at which an object may be placed in order to produce an image four times as large as the object. Draw a ray diagram and describe the nature of the image for each case.

2. A monochromatic laser beam of frequency ν_0 passes through a rotating fan blade that cuts it into a succession of wavetrains of duration τ_0. Find the frequency spectrum of the resulting finite wavetrains. Sketch the frequency spectrum, and show that its halfwidth is inversely proportional to τ_0.

D: Astrophysics

1. The Hα (6564.7 Å) lines from two stars in a binary system are observed to have maximum Doppler shifts of 0.22 Å and 0.44 Å. The period of the system is 20 years. Assume the stars are in circular orbits. What are the masses of the two stars if the line of sight lies in the orbital plane?
($G = 6.67 \times 10^{-8}$ dyne cm^2 gm^{-2}; $c = 3 \times 10^{10}$ cm s^{-1})

2. Use the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = \frac{-Gm(r)\rho}{r^2},$$

where $m(r)$ is the mass interior to r and ρ is density, to show that the pressure at the center of the star can be given by

$$P_c \sim \frac{3GM_\star^2}{4\pi R_\star^2}$$

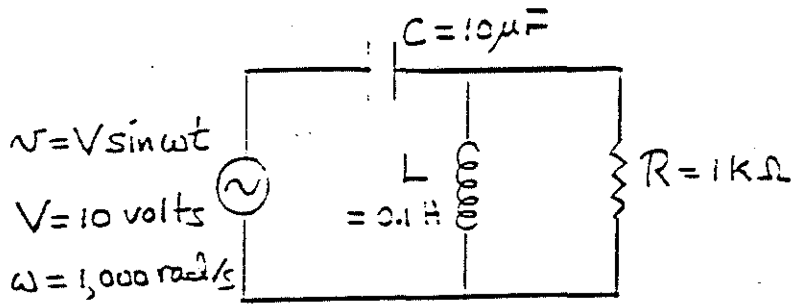
where $M_\star$ and $R_\star$ are the total mass and radius of the star, respectively. Then from the luminosity law

$$L = \frac{64\pi\sigma T^3}{3\kappa\rho} r^2 \frac{dT}{dr}$$

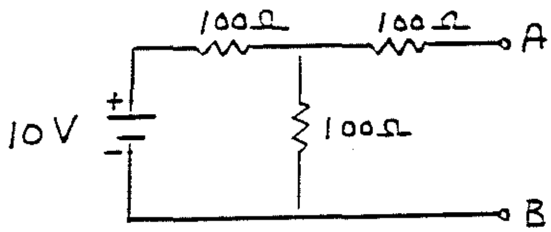
where σ is the Stefan-Boltzman constant and κ is an average specific opacity, show that the relationship between stellar mass and luminosity can be given by $L_\star \propto M_\star^3$ Note that certain approximations can be made in the derivations.

E: Electronics

1. Find the amplitude and phase angle of the voltage across the resistor in the circuit below.



2. (a) Draw both the Thevenin's equivalent and the Norton's equivalent of the circuit below.



(b) What value of resistance should one connect between terminals A and B in order that the power dissipated in the resistance be a maximum? Justify your answer.

THERMAL AND STATISTICAL MECHANICS

1. Find the specific heat C_V for a system of N three-dimensional quantum harmonic oscillators of frequency ω at a temperature T. Determine the low and the high temperature behavior of C_V.

2. (a) Explain in words why the efficiency of a heat engine is less than unity. Under what condition is a Carnot engine 100% efficient ?

   (b) Find the efficiency of a reversible heat engine operating between 600 K and 300 K. Suppose the lower temperature is further lowered to 200 K by connecting the heat engine to a reversible refrigerator which operates between 200 K and 300 K. What is the efficiency now ?

3. Consider a system of N spins in a magnetic field B along the z - direction at temperature T. Each spin can be in two directions, up and down with energies $-\mu B$ and $+\mu B$ respectively. Find the fraction of up spins, the partition function and the magnetization per spin.