Overview:

- Integrals of the form $I(\lambda) = \int g(z) e^{\lambda h(z)} dz$ as $|\lambda| \to \infty$ with $\lambda \in \mathbb{C}$
  
  - Usually two endpoints at $\infty$ and $I$ converges
  
  - Covers a huge class of problems
  
  - QFT path integral of same type

- All about the saddle points of $h(z)$ ("action") $h'(z_0) = 0$

  - Often there is a saddle point $z_0 = 0$ with $h(z_0) = 0$
    
    $\rightarrow$ perturbative saddle point

  - Other saddle points $z_i$ with $h(z_i) \neq 0$ : Instanton saddle points

  e.g. $h(z) = \frac{z^2}{2} + \frac{z^4}{4}$ $\rightarrow$ pert. saddle point at $z_0 = 0$, $h(z_0) = 0$

  $h'(z) = z + z^3 = 0$ $\Rightarrow$ $z = \pm i$

- First we can do a perturbative expansion in $\frac{1}{\lambda}$ around the pert. saddle point (e.g. in QFT), there are expansions around the instanton saddle points.

  $\rightarrow$ Which one contribute or dominate depends on the end points, STOKES PHENOM. $\rightarrow$ Instantons

We will see:

- How those expansions are calculated (Laplace Method I, Method of Steepest Descent II)

- That those expansions are usually divergent $\rightarrow$ asymptotic series

- Borel Transf. gives a different function with poles in $\mathbb{C}$

- Laplace Transf. of Borel transf. might give good solution

- In Borel plane around the poles one finds other series (around other saddle points) $\rightarrow$ RESURGENCE
Laplace's Method

Only real functions

\[ F(\lambda) = \int_{-a}^{b} e^{-\lambda R(t)} g(t) \, dt, \quad \lambda \to \infty, \lambda > 0 \]

We only focus on \( R(t) \) that has its max in \((a, b)\) and not at \( a \) or \( b \). Also take \( R''(t_{\text{max}}) < 0 \) \( \neq 0 \)

Basically the trick is that we can solve the easier integral

\[ \int_{-a}^{b} e^{-\lambda^2 t^2} \phi(t) \, dt \sim \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(0)}{2^{2n} n!} \frac{1}{\lambda^n}, \quad \lambda \to \infty, \lambda > 0 \]

Try: \( \tilde{R}(t_{\text{max}} + \tau) = -S^2 \)

\[ \rightarrow \text{solve for } T \quad T(s), \tau(0) = 0 \]

Implicit function theorem: \( \rightarrow \) does not work \( \Rightarrow \) will explain later

Solution: "Blowing up the singularity"

\[ T = SV \]

\[ \frac{\tilde{R}(t_{\text{max}} + SV)}{S^2} = -1 \]
Then: (analytic Implicit Function Theorem)

\( x_0 \in \mathbb{C}, \ f(x_0, 0) = 0, \ f(x, \varepsilon) \) analytic at \( x = x_0, \varepsilon = 0, \)

if also \( \partial_x f(x_0) \neq 0 \) \( (x_0 \text{ simple root of } f(x_0) = 0) \)

then: \( \exists \alpha > 0, \beta > 0 \text{ s.t. } \forall |\varepsilon| < \alpha \)

\( f(x, \varepsilon) = 0 \) has a unique and simple root \( x = x(\varepsilon) \)

in the disk \( |x - x_0| < \beta \).

Moreover, \( x(\varepsilon) \) is analytic for \( |\varepsilon| < \alpha, \ x(0) = x_0 \).

\[ \tilde{R}(t_{\text{max}} + \tau) = -s^2 \]

\[ f(\tau, s) := \tilde{R}(t_{\text{max}} + \tau) + s^2 \]

\[ \Rightarrow \text{find } \tau(s) \text{ in expansion in } s. \]

\[ R \sim \tau^2 \text{ close to } t_{\text{max}} \]

\[ \partial_{\tau} f(\tau, 0) = \partial_{\tau} \tilde{R}(t_{\text{max}} + \tau) \bigg|_{\tau = 0} = \alpha \partial_{\tau} \tau^2 \bigg|_{\tau = 0} = 0 \quad \text{by Implicit Function Theorem does not work.} \]

Basically the problem is that this way

\[ \tau(s) \sim \begin{cases} +s & \text{for } s \to 0 \Rightarrow s = 0 \text{ is branch point } \Rightarrow \text{not analytic there.} \\ -s & \text{for } s \to 0 \end{cases} \]
Solution:

Blowing up the singularity

\[ \tilde{R}(t_{\text{max}} + \tau) = -s^2 \]

\[ \rightarrow \tilde{R}(t_{\text{max}} + s\nu) = -s^2 \rightarrow \frac{\tilde{R}(t_{\text{max}} + s\nu)}{s^2} = -1 \]

\[ f(\nu, s) = \frac{\tilde{R}(t_{\text{max}} + s\nu)}{s^2} + 1 \]

\[ \approx \alpha \nu^2 + 1 \quad \text{for} \ |s| < 1 \]

\[ \partial_\nu f(v, 0) \bigg|_{v=0} = 2\alpha v \bigg|_{v=0} = 0 \quad \gamma \]

What's going on?

The trick is that \( s(t) = \theta(s) \) for

\[ t = s\nu \rightarrow 0 \quad \text{as} \ s \rightarrow 0, \] but that does not mean that

\[ v = 0 \quad \text{if} \]

Actually:

\[ \tilde{R}(t_{\text{max}} + s\nu) \approx \frac{1}{2} R''(t_{\text{max}}) v_0^2 + O(s^3) = -1 \]

\[ \Rightarrow v(0) = v_0 = \sqrt{-2 R''(t_{\text{max}})} \]

\[ \text{remember} \ R''(t_{\text{max}}) < 0 \]

and now

\[ \partial_\nu f(v, 0) \bigg|_{v=v_0} = 2\alpha v \bigg|_{v=v_0} \neq 0 \Rightarrow \]

\[ \Rightarrow \text{can do the change of variables} \quad \tau = t(s) = s\nu(s) \]

\[ \Rightarrow \Phi(\lambda) = \int_{-\infty}^{b} e^{\lambda R(t)} g(t) dt = \int_{-\infty}^{b} e^{\lambda \tilde{R}(t_{\text{max}} + \tau)} g(t) dt \]

\[ = e^{\lambda \tilde{R}(t_{\text{max}} + \delta)} \int_{-\alpha}^{\beta} e^{-2s^2 \lambda s^2} \gamma(s) \left( s\nu(s) + v(s) \right) ds \]

\[ \alpha = \sqrt{-\tilde{R}(t_{\text{max}} + \delta)}, \ \beta = \sqrt{-\tilde{R}(t_{\text{max}} + \delta)} \]
$$\Phi(\chi) = e^{\chi R_0^2} \int_{-\infty}^{\infty} e^{-\lambda s^2} \phi(s) \, ds$$

$$\lambda \to \infty, \lambda > 0$$

_**How to get** \( \phi^{(2n)}(0) \)

$$\phi(s) = g(t_{\text{max}} + sV(s)) (sV'(s) + V(s))$$

$$\tilde{R}(t_{\text{max}} + \tau) = -s^2, \quad \tilde{R}(0) = R(0)$$

$$\tau = sV(s)$$

$$V(0) = V_0 = \sqrt{-2 \frac{2}{R''(t_{\text{max}})}}$$

$$\phi(0) = g(t_{\text{max}}) V_0$$

$$\phi'(0) = g'(t_{\text{max}}) (V_0 + V'(0) s) V(0) + g(t_{\text{max}}) (V'(s) + V'(s))$$

$$\phi'(0) = g'(t_{\text{max}}) V_0 + 2g(t_{\text{max}}) V''(0)$$

$$\phi'(0) = \left[ g'(t_{\text{max}} + sV(s)) (V(s) + V'(s)s) (sV'(s) + V(s)) + g(t_{\text{max}} + sV(s)) (2V'(s) + sV''(s)) \right]_{s=0}$$

$$\phi''(0) = g''(t_{\text{max}}) V_0^2 + g(t_{\text{max}}) 2V'(0)$$

$$\phi'''(0) = g'''(t_{\text{max}}) V_0^3 + 6g'(t_{\text{max}}) V_0 V'(0) V(0) + 3g(t_{\text{max}}) V''(0)$$

Thus we need \( V'(0), V''(0) \)

$$\tilde{R}' = sV(s) = -s^2$$

$$\tilde{R}' = (t_{\text{max}} + sV(s))(sV'(s) + V(s)) = -2s, \quad s \to 0 \Rightarrow V(0)$$

$$\frac{d}{ds} \Rightarrow V'(0) \quad \text{and so on}$$

$$\Rightarrow \text{Mathematica}$$
Complex Integrals: Saddle Point Method

\[ F(\lambda) = \int_{C} e^{\lambda h(z)} g(z) \, dz \], \lambda \to \infty, \lambda > 0

Saddle points at \( h'(z) = 0 \Rightarrow z_i \text{ saddle points}

Turns out: \( \text{Im}(\lambda h(z)) = \text{const} \) are paths of steepest descent
(from the Cauchy–Riemann)

\[ \Rightarrow \text{at saddle points search for paths of steepest descent} \]

along this curve:

\[ F(\lambda) = e^{\lambda \text{Im}(z_i)} \int_{C_i} e^{\lambda \text{Re}(z)} g(z) \, dz \], \lambda \to \infty, \lambda > 0

real integral that can be solved with saddle Laplace's method.

Plan

- Find saddle points and their steepest descent contours \( \Gamma_i \), \( i = 0, 1 \)
- Deform original contour in sum of \( \Gamma_i \): \( C = \sum_{i} S_i \Gamma_i \)
- calc. \( F_{\Gamma_i} \) along \( \Gamma_i \) with Laplace

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Example:

\[ Z(x) = \int_{\mathbb{R}} \, dx \, e^{-\frac{1}{2} x^2 + \frac{1}{2} x^4} \]

\[ S(z) = \frac{1}{2} z^2 + \frac{1}{2} x^4 \]

\[
\begin{align*}
Z_0 &= 0 \quad \text{perturbative saddle} \Rightarrow S(z_0) = 0 \\
Z_+ &= \pm \frac{1}{\sqrt{2x}} \quad \text{instanton saddle} \Rightarrow S(z_+) = -\frac{x}{2x}
\end{align*}
\]

Steepest descent paths \( z = x + iy \) \( \Rightarrow \int S(x+iy) = \int S(z_0) \) \( \Rightarrow \) b(a) or a(b)

Show figure 2 from ubel et al.

For the expansions one gets:

\[ Z^{(0)}(x) \sim \sqrt{\frac{i}{2\pi}} \sum_{n=0}^{\infty} \frac{(\frac{2}{3})^n (4n)!}{2^{6n} (2n)! n!} x^n \quad \text{perturbative saddle point} \]

\[ Z^{(2)}(x) \sim \sqrt{\frac{i}{2\pi}} \sum_{n=0}^{\infty} \frac{(-\frac{2}{3})^n (4n)!}{2^{6n} (2n)! n!} x^n \quad \text{instanton saddle point} \]

Both diverge: \( \nu \neq \Phi_{\nu}(x) \)

\[ a_n \sim A_n^n n! \quad \text{with} \quad A = \frac{3}{2} \quad \text{instanton action} \]

Borel transform:

\[ Z(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \text{divergent} \quad x \to 0 \]

\[ B[Z](\xi) = \sum_{n=0}^{\infty} \frac{a_n \xi^n}{n!} \quad \text{finite radius of convergence} \]

\[ \int_{-\infty}^{\infty} e^{i \xi t} B[Z](\xi) \, d\xi \]

\[ \text{Laplace Transform} \]

\[ \text{Analytic continuation} \]

\[ \text{Saddle Points} \]
Borel Transform + analytic continuation

\[ B[\Phi_0](\zeta) = \frac{1}{8\sqrt{2\pi}} \frac{1}{2} + \left( \frac{5}{4}, \frac{7}{4}, \frac{11}{2} \right) \]

\[ B[\Phi_2](\zeta) = \frac{i}{8\sqrt{2\pi}} \frac{1}{2} + \left( \frac{5}{4}, \frac{7}{4}, \frac{11}{2} \right) \]

pole at \( \zeta = \frac{3}{2} = A \)

pole at \( \zeta = -\frac{3}{2} = -A \)

What do those singularities look like?

\[ \left\{ \begin{align*}
B[\Phi_0](\zeta) & \big|_{\zeta \to A} = (2) \frac{Z_0}{2\pi i (\zeta - A)} + (2) B[\Phi_1](\zeta - A) \frac{\ln(\zeta - A)}{2\pi i} + \text{holomorphic} \\
B[\Phi_1](\zeta) & \big|_{\zeta \to A} = (-1) \frac{Z_0}{2\pi i (\zeta + A)} + (-1) B[\Phi_0](\zeta + A) \frac{\ln(\zeta + A)}{2\pi i} + \text{hol.}
\end{align*} \right. \]

Go back to saddle points

The perturbative expansion around the zero-instanton, \( Z_0 = 0 \), gives \( \Phi_0 \) which is divergent. The Borel transform has a pole which, around which, one finds \( \Phi_1 \), is the expansion around the \( 2\frac{3}{2} \) instanton.
Furthermore:

The Laplace Transform (inverse Borel transform)

\[ L \]

has jumps depending on the phase of \( z \).

These jumps lead to different asymptotic expansions in different sectors of \( \mathbb{L}^2 \) which is Stokes's phenomenon. One can see the same purely from looking at the paths of steepest descent.

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