Unifying Inflation with Dark Energy in f(R) gravity & Axion DM ①

- Introduce an effective $f(R)$ gravity theory alongside a misalignment model Axion scalar field.
  - As a result, we can describe inflation, dark matter, & late-time acceleration.
  - (Model also predicts a stiff matter era preceding inflation)
- non-minimal coupling b/w axion and a f^n $G(R)$ of primordial origin.
- Will find a deSitter expansion at late times. Λ for this theory has allowed values close to existing estimate for ~~Λ_0~~ Λ_0.
- We will touch on two prerequisites before looking at the model:
  ① Basics of Axions
  ② Brief intro to $f(R)$ gravity
- Note: Assume FRW for all calculations

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① Axions

- QCD suffers from "Strong CP Problem."
  $$\mathcal{L}_{\Theta QCD} = \frac{\Theta_{QCD}}{32\pi^2}\, tr[G_{\mu\nu}\tilde{G}^{\mu\nu}]$$
  allowed in Lagrangian, but no reason why related phenomena are not found.
  $\Rightarrow$ fine-tuning problem.
- CP violating term. Gives rise to nEDM $\sim 10^{-16}\Theta_{QCD}$ e cm; Current constraints say $\Theta_{QCD} \lesssim 10^{-10}$
- Axion is a proposed ~~particle that~~ pseudoscalar coupling to $G\tilde{G}$. Dynamically sets $\Theta_{QCD} = 0$ via QCD non-perturbative effects (instantons).
  - Original Axion model introduces a complex scalar field & $\mathcal{L}$ is required to be invariant under a global $U(1)_{PQ}$ symmetry.
  - Scalar φ has a symmetry breaking potential, spontaneously broken at some scale.
  - Axion is the angular d.o.f. of this φ:
    $$\varphi = \chi e^{i\phi/f_a}$$
- Axion mass
  $$m_{a,QCD} \approx 6\times10^{-6}\,eV\left(\frac{10^{12}\,GeV}{f_a/C}\right)$$
  from Chiral P.T. mostly model independent prediction. Big $f_a \rightarrow$ DM candidate.
  - f_a - axion decay const.
  - C - "color anomaly"
- $U(1)_{PQ}$ acts as a Chiral rotation
- $V(\varphi) = \lambda\left(|\varphi|^2 - \frac{f_a^2}{2}\right)^2$

- Instantons & Axion potential

- PQ rotation on a field χ_i w/ chg $Q_{PQ,i}$

$$\chi_i \to e^{iQ_{PQ,i}\,\phi/f_a}\,\chi_i$$

- Classically, $\mathcal{L}$ is invariant under PQ rotation, but PQ rotations are anomalous at the Quantum level. This shifts the allowed term $\mathcal{L}_{QCD}$ by some amount:

$$S \to S + \int d^4x\, \frac{C}{32\pi^2}\frac{\phi}{f_a}\mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}]$$

- Chiral; use γ_5 if χ_i is a spinor
- $\mathcal{L}$ inv. implies shift symmetry $\phi \to \phi$ + constant
- $C\delta_{ab} = 2\,\mathrm{Tr}(Q_{PQ}T_aT_b)$
  T_a generators of $SU(3)$ reps. of fermions.
  "Color Anomaly"

- $\mathcal{L}_{QCD}$ doesn't affect classical EOMs, but does affect the vacuum structure, which depends on Θ_{QCD} (instantons & Θ-vacua).
  - Vacuum energy is
    $$E_{vac} \propto \cos(\Theta_{QCD}) \sim \Theta^2_{QCD}$$
  - Topologically distinct Θ-vacua means $\nexists$ transitions b/w them; E_{vac} can't be minimized. Introduction of axion, which couples to $G\tilde{G}$, means E_{vac} now depends on $(\Theta_{QCD} + C\phi/f_a)$.
  - Shift symmetry of ϕ allows us to absorb contributions to Θ_{QCD}, so
    $$E_{vac} \propto \cos(C\phi/f_a)$$
    Dependence on dynamical field ϕ
    $\Rightarrow E_{vac}$ can be minimized by EOM.

Note: Color anomaly sets # of vacua ϕ has in $[0, 2\pi f_a]$. Periodic symmetry $\Rightarrow C \in \mathbb{Z}$. $C = N_{DW}$, domain wall #.

- QCD instantons generate axion potential:

$$V(\phi) = m_u \Lambda^3_{QCD}\left[1 - \cos\left(\frac{N_{DW}\phi}{f_a}\right)\right]$$

where cosine dependence comes from Θ_{QCD} dependence of E_{vac} in lowest order instanton calculation.

- Λ_{QCD} is the QCD confinement transition scale

$\Rightarrow$ non-pert. effects switch on at some scale & induce potential / mass
↳ shift symmetry broken to discrete symm. $\phi \to \phi + 2\pi f_a/C$

Quick recap:

- Global $U(1)_{PQ}$ symmetry for classical action spontaneously broken at some scale f_a, leads to angular d.o.f. ϕ/f_a w/ a shift symmetry
- $U(1)_{PQ}$ symmetry is anomalous. Explicit breaking generated by quantum effects (instantons) at some scale Λ_a (non-perturbative effects scale)
- ϕ being angular means quantum effects must respect the residual symmetry $\phi \to \phi + 2n\pi f_a$. Thus axions obtain a periodic potential when non-perturbative effects switch on.

- $V(\phi) = \Lambda_a^4 \left[1 - \cos\left(\frac{N_{DW}\phi}{f_a}\right)\right]$

- simple choice for axion potential. Chosen $U(x)$ minimum @ $x = 0$.

- A model-independent approach to studying axions is possible assuming small displacements $\phi \ll f_a$

$$V(\phi) \approx \frac{1}{2} m_a^2 \phi^2 \quad , \quad m_a^2 = \frac{\Lambda_a^4}{f_a}$$

- Λ_a usually $< f_a$, making m_a parametrically small.

- Cosmological Axion Field

- For a minimally coupled real scalar field in GR, we have

$$S_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\partial\phi)^2 - V(\phi) \right]$$

- only valid after sym. breaking and non-pert. effects switched on

- Varying wrt ϕ yields the EOM

$$\Box\phi - \frac{\partial V}{\partial \phi} = 0 \; ; \; \Box = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \right)$$

- In turn, this yields

$$\ddot{\phi} + 3H\dot{\phi} + m_a^2 \phi = 0$$

- We now want to consider Axions with the phenomenology of the Misalignment Model

- Simply put, misalignment corresponds to the scenario where the axion field has a coherent initial displacement.
  - ~~We'll~~ Later we'll see how this determines cosmological behavior of Axions.

- In short, axion transitions from cosm. const-like behavior to oscillating around $w=0$ and behaving more like ordinary matter.

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② f(R) Gravity

- Historical Motivation

- Nonrenormalizability of GR
  - Calculations suggested that gravity may need to be supplemented by higher-order curvature invariants.
  - ΛCDM, despite successes, still somewhat lacking; can be thought of as an empirical fit to data, less theory motivated

- Can also address cosmological accel. w/o Dark Energy

- Basic Idea: generalize $\mathcal{L}$ in the Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa}\int d^4x \sqrt{-g}\, R$$

$$\hookrightarrow S = \frac{1}{2\kappa}\int d^4x \sqrt{-g}\, f(R)$$

with $f(R)$ some fn of the Ricci Scalar.

- Despite possibly hand-wavy motivation/~~questions~~ reasons as to why, R over other invariants, it is one of the most straightforward ways to tackle (modified) gravity.
- That being said, it's still closer to a toy theory, useful for exploring principles & limitations of modified gravity.

- Choose R and not other invariants like $R_{\mu\nu}R^{\mu\nu}$ because
  ① They're simple + still cover basics of higher-order gravity
  ② ~~Best~~ Despite being an effective theory, and despite concerns about where corrections would have relevant effects, observed data corresponding to some energy scale implies inevitability that some parameter(s) or result(s) will appear "unnatural" (wording?)

- Action & Field Eqns

- 3 ~~principles~~ variational principles

  1. Metric - standard principle
  2. Palatini - metric & connection are assumed indep. & both varied, assuming the matter action indep. of connection
  3. Metric-Affine - Palatini but w/o assumption of matter action indep. of connection.

- Interestingly, possible to obtain

- Each formalism has their own benefits & drawbacks. For our purposes, we only need the metric formalism.

- Starting with a full action

$$S = \frac{1}{2\kappa}\int d^4x \sqrt{g}\, f(R) + S_M(g_{\mu\nu}, \psi)$$

- ψ stands for all matter fields

Varying wrt the metric yields

$$f'(R) R_{\mu\nu} - \tfrac{1}{2} f(R) g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box] f'(R) = \kappa T_{\mu\nu}$$

- $T_{\mu\nu} = \frac{-2}{\sqrt{g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$

modulo surface terms.

- Note: For S_{EH}, surface terms are a total variation of some quantity, but not so for $f(R)$. Since S has higher order derivatives of the metric, it should be possible to fix more d.o.f. on the boundary than just those of the metric.

- as R contains 2^{nd} derivatives of the metric, there are 4^{th} order pde's. For $f(R) \sim R$, we re-obtain standard GR.

- Regarding maximally symmetric sol$^{\underline{ns}}$:

  Trace of above eqns yields

$$f'(R) R - 2f(R) + 3\Box f' = \kappa T, \quad T = g^{\mu\nu} T_{\mu\nu}$$

  for R constant, $T=0$, one can obtain

  $R=0$: Minkowski

  $R=C$: (anti) de Sitter

(*) $T=0 \not\Rightarrow R=0$ or ~~const~~ constant

• f(R) Gravity with Axion DM

- Consider a vacuum f(R) grav. thy. with an axion DM scalar field. Non-minimal coupling b/w scalar ϕ & higher powers of R:

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2} f(R) + \frac{1}{2\kappa^2} h(\phi) G(R) - \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - V(\phi)\right]$$

- for simplicity, let

$$\mathcal{F}(R,\phi) = \frac{1}{\kappa^2} f(R) + \frac{1}{\kappa^2} h(\phi) G(R)$$

- Varying wrt the metric (using metric formalism), we find two equations

$$3H^2 F = \frac{1}{2}\dot{\phi}^2 + \frac{RF - \mathcal{F} + 2V}{2} - 3H\dot{F}$$

$$-3FH^2 - 2\dot{H}F = \frac{1}{2}\dot{\phi}^2 - \frac{RF - \mathcal{F} + 2V}{2} + \ddot{F} + 2H\dot{F}$$

where $F = \frac{\partial \mathcal{F}}{\partial R}$

- Varying wrt ϕ, the axion EOM is then

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2}\left(-\mathcal{F}'(R,\phi) + 2V'(\phi)\right) = 0$$

with prime meaning $\frac{\partial}{\partial \phi}$.

- The choice for f(R) gravity is

$$f(R) = R + \frac{1}{36 H_i^2} R^2$$

- akin to well-known R^2 or Starobinsky models

- The nonminimal coupling choice depends on axion phenomenology & late time behavior. Assume

$$h(\phi) \sim \frac{1}{\phi^\delta},\ \delta > 0;\quad G(R) \sim R^\gamma,\quad 0 < \gamma < 0.75$$

• Description of Inflation & Dark Energy

- Recall earlier discussion of Axion properties. In addition, ~~to~~ we require one more assumption:

$$2V'(\phi) \gg \mathcal{F}'(R,\phi) \quad \text{or} \quad 2V(\phi) \gg \frac{1}{\kappa^2} h(\phi) G(R)$$

The choices for $h(\phi)$ & $G(R)$ ~~m~~ satisfy this constraint on the dynamics of ϕ.

← Axion field evolution

- The assumption means ϕ EOM. is mainly affected by $V(\phi)$ during/after inflation. Thus we can use misalignment pheno.

- We consider the relationship b/w H and m_a as the Universe evolves
  - During inflation, $H \gg m_a$ and the potential is $V(\phi) \simeq \frac{1}{2} m_a^2 \phi_i^2(t)$.

    The field is overdamped & the init. conditions $\dot{\phi}(t_i) = \delta \ll 1$, $\phi(t_i) = f_a \theta_a$ hold.

    So ϕ is frozen during inflation & contributes a small cosm. const term.

- i denotes the cosmological era
- t_i is cosmic time during inflationary period
- θ_a is initial misalignment angle.

- Numbers:

$H_I = \mathcal{O}(10^{13})$ GeV for low-scale infl. scenario

$f_a \sim \mathcal{O}(10^{11})$ GeV

$\theta_a \sim \mathcal{O}(1)$

$m_a \sim \mathcal{O}(10^{-12})$ eV

} Pheno. reasons. Most plausible values

- H_I from Planck + BICEP 2
- θ_a is init. misalignment angle.

Implying

$$2\kappa^2 V(\phi) \sim \mathcal{O}(3 \times 10^{-38}) \text{ eV}$$

while for $\delta \sim \mathcal{O}(3)$, $0 < \gamma < 0.75$,

$$h(\phi) G(R) \sim \mathcal{O}(2 \times 10^{-88}) \text{ eV}$$

so the req'd constraints still hold.

- As $m_a \sim H$, axion field oscillation begins & continues to $m_a \gg H$, and we seek slowly ~~varying sol~~ oscillating sol^{n}'s

$$\phi(t) = A(t)\cos(m_a t)$$

Plugging into EOM and working to leading order in ε, we obtain

$$-\frac{2\dot{A}(t)\sin(m_a t)}{m_a} - \frac{3A(t)H(t)\sin(m_a t)}{m_a} = 0$$

$$\xrightarrow{H = \frac{\dot{a}}{a}} \quad \frac{\partial A}{A} = -\frac{3}{2}\frac{\partial a}{a} \Rightarrow A \sim a^{-3/2}$$

- ~~$\dot{\rho}$~~ given the ~~density~~ energy density

$$\rho_a \sim \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m_a^2\phi^2$$

we find to leading order

$$\boxed{\rho_a \sim A^2 \sim a^{-3}}$$

CDM Dynamics after inflation

$\Rightarrow$ - Evolution of $H(t)$ does not depend on ϕ.
- $f(R)$ alone determines dynamics of spacetime, with ϕ as DM.
- Sinusoidal time dep. of ϕ for $H \ll m_a$ means ~~$\langle w_a \rangle \sim 0$~~ $\langle w_{eff} \rangle \sim 0$ for ϕ, indep of $H(t)$.

- $\frac{\dot{A}}{m_a} \sim \frac{H}{m_a} \sim \varepsilon \ll 1$

Satisfied by $A(t)$

- $f(R)$ Gravity's Role in the model

- Consider the first of the two Friedmann eqs. for R^2 gravity, it becomes

$$3H^2\left(1+\frac{1}{18H_i^2}R + h(\phi)G'(R)\right) \simeq$$

$$\frac{1}{2}K^2\dot{\phi}^2 + \frac{1}{72H_i^2}R^2 + 2VK^2 - \frac{1}{6H_i^2}H\dot{R}$$

$$- 3H\dot{R}\,G''(R)\,h(\phi)$$

- During inflation, $R \sim 12H_I^2 \sim \mathcal{O}(1.45\times10^{45})$ eV, so $G'(R) \sim R^{\gamma-1}$ is highly suppressed (LHS). Same for $G''(R)$ on RHS.

- For $\frac{1}{2}K^2\dot{\phi}^2$, recall $\dot{\phi}(t_i) \ll 1$. for $\phi(t_i) \sim 10^{-10}$,

$$\frac{1}{2}K^2\dot{\phi}^2 \sim \mathcal{O}(8\times10^{-74})\,eV^2$$

$$2VK^2 \sim \mathcal{O}(3\times10^{-38})\,eV^2$$

$$\frac{1}{72H_i}R^2 \sim \mathcal{O}(2\times10^{62})\,eV^2$$

$$\Rightarrow\ 3H^2\left(1+\frac{1}{18H_i^2}R\right) \simeq \frac{1}{72H_i^2}R^2 - \frac{1}{6H_i^2}H\dot{R}$$

using $R = 12H^2 + 6\dot{H}$ for FRW, reduces to

$$\ddot{H} - \frac{\dot{H}^2}{2H} + 3H_i^2 H = -3H\dot{H}$$

$$\xrightarrow{\text{slow Roll}} 3H_i^2 H = -3H\dot{H}$$

$$\Rightarrow H(t) = H_0 - H_i^2 t$$

-> Quasi - DeSitter evolution

- We see (approx) independence of inflationary evolution from axion behavior/evolution.

(Aside: maybe quick discussion on R^2 cosmology behavior/characteristics, time allowing)

– Dark Energy / Late time Era

– In this ~~regime,~~ era, R is very small. Friedmann eq. becomes approx:

$$3H^2 h(\phi)\, G'(R) \simeq -3H\dot{R}\, G''(R)\, h(\phi)$$

$$\hookrightarrow \; RH \simeq (1-\gamma)\dot{R}$$

– neglect the term $\sim H^3$, as it's subdominant at late times

$$\Rightarrow H(t) \simeq \frac{\sqrt{2(1-\gamma)}\sqrt{\Lambda}}{\sqrt{3-4\gamma}} \tanh\left[\frac{1}{2}\left(\frac{\sqrt{2(3-4\gamma)}\,\Theta\sqrt{\Lambda}}{\sqrt{1-\gamma}} + \frac{\sqrt{2(3-4\gamma)}\,\sqrt{\Lambda}\,t}{\sqrt{1-\gamma}}\right)\right]$$

– Λ & Θ are integration constants

$$\hookrightarrow H(t_\infty) \simeq \frac{\sqrt{2(1-\gamma)}\sqrt{\Lambda}}{\sqrt{3-4\gamma}} \quad ☆$$

is approximately constant at late times

– deSitter evolution i.e., accelerating expansion

– Accelerating evolution also seen in the form of the deceleration parameter $q = -1 - \dot{H}/H^2$:

$$q = -1 - \frac{(3-4\gamma)\,\mathrm{csch}^2\left[\frac{1}{2}\left(\frac{\sqrt{2(3-4\gamma)}}{\sqrt{1-\gamma}}\left(\Theta\Lambda + \sqrt{\Lambda}\,t\right)\right)\right]}{2(1-\gamma)}$$

– negative as t is large, thus implying expansion.

– $w_{eff} = -1$ as well; DE-like E.O.S.

☆ Axion coupling $h(\phi)\, G''(R)$ thus affects late-time evolution in a dominant way, specifically ~~controlling~~ leading to unnatural acceleration, providing a Dark Energy Era.

- In a phenomenological context, consider that today $H_0 \sim 10^{-33}$ eV. for $\gamma \approx 0.74$,

$$\Lambda \sim 7.7 \times 10^{-68}\ eV^2$$

$$\gamma = 0.2 \rightarrow \Lambda \sim 1.3 \times 10^{-66}\ eV^2$$

Generally:

$$0 < \gamma < 0.74, \quad 1.5 \times 10^{-66}\ eV^2 < \Lambda < 7.69 \times 10^{-68}\ eV^2$$

close to observed value of $\sim 10^{-66}\ eV^2$

- Λ much smaller as $\gamma \rightarrow 0.75$

- Possibility of equation ☆ is the relation b/w H_0 and Λ? Interesting, given $\Lambda_0 \sim H_0^2$ for present-day cosm. constant.

(Note: Similar phenomenology from a pure $f(R)$ model w/o non-minimal coupling

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \frac{1}{36 H_i^2} R^2 - \delta R^\gamma\right) - \mathcal{L}_{\phi,\text{free}}\right]$$
)

• Summary

- $f(R)$, R^2 gravity + Axion scalar field effective theory. Unifies inflation era w/ DE era, describes DM.
  - Axion frozen to primordial vev at early times; Starobinsky model $\rightarrow$ quasi-deSitter evolution
  - axion oscillation + slow-varying evolution, $\rho_a \sim a^{-3}$ $\rightarrow$ DM behavior, indep. of background Hubble rate
  - At late times, $h(\phi) G(R)$ dominates, $H \sim$ de Sitter $\rightarrow$ Cosm. const + accelerating universe.
- Phenomenological considerations make this effective thy. ~~a possibility~~ an interesting possibility.