General Solution to the U(1) and U(1) x U(1) Anomaly Equations

It has been long known that anomalies restrict the matter content in particle physics. In particular, the ratios of the U(1) hypercharges of the standard model are almost fixed by the cancellation of anomalies.

In this seminar we will focus only the gauge groups of U(1) and U(1) x U(1), not the standard model gauge group.

Let's first consider U(1). The types of anomalies encountered are U(1) anomalies and mixed U(1) gravitational anomalies as in the figures below:

![Diagram](a) U(1) anomaly

![Diagram](b) U(1) x U(1) anomaly

Suppose we have n left-handed chiral fermions of charge $z_i$. Then figure a. leads to anomaly equation of the form

$$\sum_{i=1}^{n} z_i = 0 \quad - (i)$$

and figure b leads to

$$\sum_{i=1}^{n} z_i = 0 \quad - (ii)$$

Note that (ii) is required as long as our theory is placed on a curved space-time for mathematical consistency. Also there can be more anomaly equations if we include additional gauge groups as we will see for the case U(1) x U(1).

Also note that equations (i) & (ii) will not cause any problems if we work only with Dirac fermions, and with gauge fields which are coupled in the same manner to both left- and right-handed fermions. Such theories are called vector-like, which we are not interested in. We then will assume that we have removed all vector-like matter so that the left-handed and right-handed fermions have no charges in common.

The requirement that $z_i$'s have to be rational (or integer after multiplying with a suitable number) numbers is basically from the fact that any U(1) gauge group must be embedded in a non-abelian structure in order to ensure that its gauge boson is well-behaved at high energies.
Solving equations (1) and (2) now becomes a Diophantine type which can be solved only numerically (for n = 5 cases) as we will show below that why the case n = 4 is not interesting (or not allowed).

For n = 2 it is clear that \( x_1 = -x_2 \) which is vector-like and nothing is interesting.

For n = 3, eqn (1) has no solution due to the famous Fermat theorem in the number theory.

For n = 4 we have

\[ x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \quad \text{and} \quad x_1 + x_2 + x_3 + x_4 = 0. \]

Write \( x_4 = -x_1 - x_2 - x_3 \) and substitute in the first equation to get

\[ x_1^3 + x_2^3 + x_3^3 - (x_1 + x_2 + x_3)^3 = 0, \quad \text{or} \]

\[ (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = 0. \]

Thus, two of the three fermions must have charges of opposite sign and equal magnitude, say \( x_1 = -x_2 \) which then also force \( x_3 = -x_3 \). This set of solution is again vector-like. It turns out that \( n \) must be at least 5 in order for the solution to be not vector-like. We will call this set as “chiral set.” Usually numerical method is used to find such sets.

Until recently a group of researchers in ref [1] found a method to solve this particular problem exactly for \( n = 5 \).

What they did is first identify the vector-like set of solution in each of eqn (1) and (2) in a particular form.

We will consider the case when \( n \) is odd and when \( n \) is even separately as the the way to construct the vector-like sets is qualitatively different.

i) \( n \) even (\( \geq 6 \))

We construct the vector-like set \( \{ \vec{x}_n \} \) of (1) by

\[ \{ \vec{x}_1 \} = \{ k, k_1, \ldots, k_n, -k, -k_1, \ldots, -k_n \} \]

and of (2) by

\[ \{ \vec{x}_2 \} = \{ 0, 0, k_1, \ldots, k_n, -k_1, \ldots, -k_n \} \]

where \( m = \frac{n}{2} - 1 \geq 2 \), \( k_1, k_2 \) are integer.

Then the chiral sets that satisfy both (1) and (2) will be obtained by

\[ \{ \vec{x}_n \} = \{ \vec{x}_1 \} \oplus \{ \vec{x}_2 \} \]

where \( \oplus \) is called “merge” operation defined as

\[ \{ \vec{x}_n \} \oplus \{ \vec{y}_n \} = (\sum_{i=1}^{n} x_i y_i) \{ \vec{x}_n \} - (\sum_{i=1}^{n} x_i y_i) \{ \vec{y}_n \} \]
Explicitly the chiral set \( \{ \vec{z} \} \) is

\[
\{ \vec{z} \} = \{ \ell_1 \vec{s}_1, \ell_2 \vec{s}_2, \ell_3 \vec{s}_3 + \ell_4 \vec{s}_4, \ldots, \ell_m \vec{s}_m + \ell_m \vec{s}_m \}
\]

where

\[
\vec{s}_i = \sum_{i=1}^{m-1} (\ell_{i+1} - \ell_i) \vec{e}_i - (\ell_i + k_i) \vec{e}_m \quad \text{and} \quad \vec{s}_m = k_1 \vec{e}_1 + \sum_{i=2}^{m-1} k_i (\ell_i - \ell_{i-1}) - \ell_i \vec{e}_m
\]

Note that if \( \{ \vec{z} \} \) is a chiral set then \( c \{ \vec{z} \} \) (\( c \in \mathbb{Z} \)) is also a chiral set. It is therefore sufficient to consider coprime sets of charges i.e., sets where the greatest common divisor (gcd) of the \( n \) charges is 1. And we will call such sets as primitive solutions to the anomaly equations. The table below show such solutions for our \( n=6 \) case.

\[
\begin{align*}
\vec{z}_1 &= \ell_1 (\ell_1^2 (k_2 - k_1) - \ell_1^2 (k_1 + k_2)) \\
\vec{z}_2 &= \ell_2 (\ell_1 (k_2 - k_1) - \ell_1^2 (k_1 + k_2)) \\
\vec{z}_3 &= \ell_3^2 k_1 (k_1 - k_2) - \ell_2 (\ell_1 + k_1) (\ell_1^2 - \ell_1 k_2 + k_1 \ell_2) \\
\vec{z}_4 &= \ell_4 (\ell_2 (\ell_1 + k_2) + (k_2 - k_1) (k_2 \ell_2 - \ell_1 k_2 + k_2 \ell_2)) \\
\vec{z}_5 &= \ell_5^2 k_2 (k_1 + k_2) - \ell_1 (k_1 - k_2) (\ell_1^2 + k_1 \ell_2 + \ell_2 \ell_4) \\
\vec{z}_6 &= \ell_6^2 (k_1 - k_2) + \ell_1 (\ell_1 + k_2) (\ell_1^2 - \ell_1 k_1 + k_2 \ell_2)
\end{align*}
\]

<table>
<thead>
<tr>
<th>Primitive solution ( { \vec{z} } )/gcd(( \vec{z} ))</th>
<th>( (k_1, k_2, \ell_1, \ell_2) )</th>
<th>gcd(( \vec{z} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{5, -4, -4, 1, 1, 1}</td>
<td>( (1, -2, 1, 2) )</td>
<td>1</td>
</tr>
<tr>
<td>{6, -5, -5, 3, 2, -1}</td>
<td>( (2, 0, 1, -1) )</td>
<td>1</td>
</tr>
<tr>
<td>{11, -9, -9, 4, 4, -1}</td>
<td>( (2, 3, 2, -2) )</td>
<td>8</td>
</tr>
<tr>
<td>{11, -9, -9, 5, 1, 1}</td>
<td>( (1, 3, 1, -1) )</td>
<td>2</td>
</tr>
<tr>
<td>{11, -10, -8, 5, 4, -2}</td>
<td>( (-1, 2, 2, -1) )</td>
<td>2</td>
</tr>
<tr>
<td>{12, -11, -10, 8, 6, -5}</td>
<td>( (3, 2, 2, 3) )</td>
<td>10</td>
</tr>
</tbody>
</table>

TABLE I. Primitive solutions to the anomaly equations for 6 fermions in canonical form for \( z_1 \leq 12 \). A choice of the \( k_1 \), \( k_2 \), \( \ell_1 \), \( \ell_2 \) parameters and the greatest common divisor that generate the primitive solution from the set (17) are shown.
a) $n$ odd ($n \geq 5$)

In this case we start with two vector-like sets of the form
\[
\{\tilde{u}_+\} = \{0, k_1, ..., k_{m+1}, -k_1, ..., -k_{m+1}\} \quad \text{and} \quad \{\tilde{u}_-\} = \{l_1, ..., l_m, k_1, -l_1, ..., -l_m, -k_1\},
\]
where $m = \frac{n-3}{2}$ and $k_1, l_1$ are integers.

Likewise we can construct chiral sets that satisfy the anomaly equations by using the merger operation which is
\[
\{\tilde{u}\} = \{\tilde{u}_+\} \oplus \{\tilde{u}_-\} \quad \text{or more explicitly}
\]
\[
\{\tilde{u}\} = \left\{ l_1 s_-, k_1 s_+ + l_2 s_-, ..., k_{n+1} s_+ + l_m s_-, k_1 s_+ + k_1 s_-, ..., -k_1 s_- - k_1 s_-, -k_{n+1} s_- - k_m s_- \right\}
\]
where
\[
s_+ = \sum_{i=1}^{\frac{n-1}{2}} k_i (l_{i+1} - l_i) + k_1 (k_1 - l_m) - k_{n+1} k_1,
\]
\[
s_- = \sum_{i=1}^{\frac{n-1}{2}} k_i (l_i - k_{i+1}) + k_1 (l_m - k_1) + k_{n+1} k_1.
\]

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**Geometric Approach**

A year later the authors in ref [2] use a geometric approach to understand the previous results. They can explain how the merger operator appears by using Projective Geometry which we are going to discuss now.

We can combine the anomaly equations (i) and (ii) as
\[
\sum_{i=1}^{n-1} \tilde{z}_i^3 = \left( \sum_{i=1}^{n-1} \tilde{z}_i \right)^3 \quad - (5) \quad \text{where} \quad \tilde{z}_n = - \frac{1}{2} \sum_{i=1}^{n-1} \tilde{z}_i \quad - (4).
\]

Then we can look at (5) as a cubic hypersurface in a projective space $\mathbb{P}^{n-2}$ where $\mathbb{Q}$ is an associated Euclidean space with only rational numbers.
Then we can construct two projective points which lie on this cubic surface as follow

For $n$ even
\[
\Gamma_1 = \left[ k_1 : k_2 : k_3 : \ldots : k_m : k_{m+1} : -k_1 : -k_2 : -k_3 : \ldots : -k_m \right] \quad m = \frac{n-1}{2}
\]
\[
\Gamma_2 = \left[ 0 : k_2 : k_3 : \ldots : k_m : -k_2 : -k_3 : \ldots : -k_m \right]
\]

For $n$ odd
\[
\Gamma_1 = \left[ k_1 : k_2 : k_3 : \ldots : k_{m+1} : -k_1 : -k_2 : \ldots : -k_{m+1} \right] \quad m = \frac{n-1}{2} - 1
\]
\[
\Gamma_2 = \left[ k_2 : k_3 : \ldots : k_m : k_{m+1} : 0 : -k_2 : \ldots : -k_m : -k_{m+1} \right]
\]

Then the third (chiral) projective point that also lies on the surface can be found from
\[
\Gamma_3 = \alpha_1 \Gamma_1 + \alpha_2 \Gamma_2 \quad \text{where } \alpha_1, \alpha_2 \text{ are integers to be determined}
\]

We then substitute $\Gamma_3$ in the surface equation
\[
\sum_{i=1}^{n-1} z_i^3 = \left( \sum_{i=1}^{n-1} z_i \right)^3
\]
and after doing some math we can get
\[
[\alpha_1 : \alpha_2] = \left[ \sum_{i=1}^{n-1} \Gamma_i^6 \ell_i^2 : -\sum_{i=1}^{n-1} \Gamma_i^2 \ell_i^4 \right] \quad \text{where } \ell_i = (\Gamma_i^a) - \left( \sum_{j=1}^{n-1} \Gamma_j^a \right) ^2
\]
or in other words
\[
\Gamma_3 = \left( \sum_{i=1}^{n-1} \Gamma_i^6 \ell_i^2 \right) \Gamma_1 - \left( \sum_{i=1}^{n-1} \Gamma_i^2 \ell_i^4 \right) \Gamma_2
\]

which almost looks like the result from merger operation.
Solution for $U(1) \times U(1)$ anomaly equations

In addition to the anomaly equations (1) and (2) for each $U(1)$ gauge group, we have 2 more equations to be satisfied which arise from triangle diagrams with two external gauge bosons of one kind and a single external gauge boson of the other type:

\[
\sum_{i=1}^{n} Z_{ji}^2 Z_{j' i} = 0 \quad - (5) \quad \text{and} \quad \sum_{i=1}^{n} Z_{ji} Z_{j' i} = 0 \quad - (6) \quad ; \quad 1 \leq j, j' \leq n
\]

Here we use the notation that $Z_{ji}$ is the charge of the $i$th particle under the $U(1)$ gauge group and $Z_{ji}$ is under the $U(1)'$ gauge group which collectively can be written as

\[
[\bar{Z}, \bar{Z}] = \left( \begin{array}{c|c} \bar{Z}_1 & \bar{Z}_2 \\ \bar{Z}_3 & \bar{Z}_4 \\ \vdots & \vdots \\ \bar{Z}_m & \bar{Z}_n \end{array} \right)
\]

It turns out that the minimum number of fermions that makes $[\bar{Z}, \bar{Z}]$ satisfy all the anomaly equations (6 equations) is $n = 6$ and we will consider only for this case.

In order to solve this we first obtain the sets of solution for $\bar{Z}_i$ and we require it to be chiral sets. Thus we can use the solution that we found from the first section for our $\bar{Z}_i$. 
It is shown in ref([1]) that the sets of solution for \( \hat{\zeta} \) (together with \( \hat{\xi} \)) that satisfy all the anomaly equations can be constructed entirely by the knowledge of \( \hat{\zeta}_1 \). One of the solution is

\[
\begin{bmatrix}
\hat{\zeta}_1 \\
\hat{\zeta}_2 \\
\hat{\zeta}_3 \\
\hat{\zeta}_4 \\
\hat{\zeta}_5 \\
\hat{\zeta}_6
\end{bmatrix} = \begin{bmatrix}
\hat{\xi}_1 \\
\hat{\xi}_2 \\
\hat{\xi}_3 \\
\hat{\xi}_4 \\
\hat{\xi}_5 \\
\hat{\xi}_6
\end{bmatrix} \begin{bmatrix}
KV_1 + \alpha \xi_1 \\
-KV_1 + \alpha \xi_1 \\
KV_3 + \alpha \xi_3 \\
-KV_3 + \alpha \xi_3 \\
KV_5 + \alpha \xi_5 \\
-KV_5 + \alpha \xi_5
\end{bmatrix}
\]

where

\[
\begin{align*}
\psi_1 &= (\xi_{15} + \xi_{16}) \left[-(\xi_{11} - \xi_{14}) (\xi_{15} - \xi_{16}) + \left(\xi_{15} - \xi_{14}\right)^2\right] \\
\psi_2 &= -(\xi_{13} + \xi_{14}) \left[(\xi_{15} - \xi_{14}) (\xi_{15} - \xi_{16}) + \left(\xi_{15} - \xi_{14}\right)^2\right] \\
\psi_3 &= (\xi_{11} + \xi_{16}) \left[(\xi_{15} - \xi_{14}) (\xi_{15} - \xi_{16}) + \left(\xi_{15} - \xi_{14}\right)^2\right]
\end{align*}
\]

and \( K, a \) are arbitrary integers.

For example if we pick the first set of \( \hat{\zeta}_1 \) in the table 1 where

\[
[\hat{\zeta}_{11}, \hat{\zeta}_{13}, \hat{\zeta}_{14}, \hat{\zeta}_{15}, \hat{\zeta}_{16}] = [5, -4, -9, 1, 1, 1]) \quad \text{and} \quad a=0, \quad K=1
\]

we obtain

\[
\begin{bmatrix}
\hat{\zeta}_1 \\
\hat{\zeta}_2 \\
\hat{\zeta}_3 \\
\hat{\zeta}_4 \\
\hat{\zeta}_5 \\
\hat{\zeta}_6
\end{bmatrix} = \begin{bmatrix}
5 \\
-4 \\
1 \\
1 \\
2 \\
0
\end{bmatrix}
\]

If we introduce 6 fermions \( \psi_1, \psi_2, ..., \psi_6 \) having charges under \( U(1)_x \times U(1)_y \) as above and also introduce a scalar \( \phi \) having charge (-20), then we are allowed to write the following Lagrangian interactions in 4 dimensions.

\[
L \supset -\psi_1 \phi^\dagger \psi_5 - \frac{g_1}{M_3} \phi^\dagger \psi_1 \psi_6 - \frac{g_2}{M_3} (\phi^\dagger)^4 \psi_1 \psi_5 + h.c
\]