Relativistic Imperfect fluid

* Perfect fluid

A perfect fluid have a same fluid velocity v at every point, such that an observer moving with v finds ~~d~~ the fluid perfectly isotropic around it.

In fluid rest frame the components of energy momentum tensors are

$$\tilde{T}^{00} = \rho, \quad \tilde{T}^{ij} = p\delta^{ij}, \quad \tilde{T}^{i0} = \tilde{T}^{0i} = 0$$

In the general frame $x^\alpha = \Lambda^\alpha{}_\beta \tilde{x}^\beta$

$$T^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \tilde{T}^{\mu\nu}$$

The components are

$$T^{00} = \gamma^2(\rho + pv^2)$$

$$T^{ij} = p\delta_{ij} + (p+\rho)\gamma^2 v_i v_j$$

$$T^{i0} = (p+\rho)\gamma^2 v_i$$

Which can be written in the covariant form as follows:

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)u^\alpha u^\beta$$

where $u^\alpha = (\gamma, \gamma v^i)$, $\Rightarrow$ $u_\alpha u^\alpha = -1$

In addition, there would be extra conserved quantity in the fluid. Let n be the density of the conserved quantity, the current in the fluid rest frame is given by

$$\tilde{J}^\mu = (\tilde{J}^0, \tilde{J}^i) = (n, 0)$$

Under the general frame

$$J^\alpha = \Lambda^\alpha{}_\nu \tilde{J}^\nu$$

we can write

$$J^\alpha = (J^0, J^i) = (\gamma n, \gamma n v^i) = n u^\alpha$$

The current and energy-momentum tensor conserved in this simple set up.

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0, \qquad \frac{\partial J^\alpha}{\partial x^\alpha} = 0$$

* Relativistic imperfect fluid

In perfect fluid the mean free path and time between the collision are so short that the isotropy is maintained. But if there is pressure, density or velocity fluctuation of the order of mean free path or mean free time or both, the equilibrium is broken the part of K.E. is dissipated into heat.

In effect it ~~has~~ modifies the current and the energy momentum tensor in the gradient expansion.

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)u^\alpha u^\beta + \tau^{\alpha\beta}$$

$$J^\alpha = n u^\alpha + j^\alpha$$

Let ρ and n be the energy density and particle number in comoving frame

$$T^{00} = \rho, \quad J^0 = n$$

and comoving frame is the one in which

$$u^\alpha = (u^0, u^i) = (1, 0)$$

Other ~~the~~ parameter, like p are dependent on ρ, n and u^α.

There is ambiguity in the defn of u^α: weather it is velocity of energy transport or that of particle transport. Two are not the same in the case of dissipation.

Landau: $u^\alpha \rightarrow$ energy transport $\Rightarrow T^{i0} = 0$

Eckart: $u^\alpha \rightarrow$ particle transport $\Rightarrow J^{i} = 0$

We adopt Eckart frame

$$\tau^{00} = J^0 = J^i = 0 \Rightarrow \text{Comoving frame}$$

In general Lorentz frame

$$u^\alpha u^\beta \tau_{\alpha\beta} = 0, \quad J^\alpha = 0$$

All the effect of dissipation now shows up in $\tau_{\alpha\beta}$ in this frame. Our task is to construct $\tau_{\alpha\beta}$. We use entropy current argument to construct it.

$\tau^{\alpha\beta}$

start with $$u_\alpha \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0$$

$$u_\alpha \frac{\partial p}{\partial x^\alpha} + \underbrace{u_\alpha \frac{\partial}{\partial x_\beta}\left[(p+\rho)\, u^\alpha u^\beta\right]}_{-\frac{\partial}{\partial x_\beta}\left[(p+\rho)u^\beta\right] - (p+\rho)u^\beta u^\alpha \underbrace{\frac{\partial u_\alpha}{\partial x^\beta}}_{0}}$$

$$= u^\beta \frac{\partial p}{\partial x^\beta} - \underbrace{\frac{\partial}{\partial x^\beta}\left[(p+\rho)\, u^\beta\right]}_{(p+\rho)\frac{\partial u^\beta}{\partial x^\beta} + u^\beta \frac{\partial}{\partial x^\beta}(p+\rho)}$$

$$\boxed{\frac{\partial}{\partial x^\alpha}(n u^\alpha) = 0 \Rightarrow n\frac{\partial u^\alpha}{\partial x^\alpha} + u^\alpha \frac{\partial n}{\partial x^\alpha} = 0}$$

$$(p+\rho)\left(-\frac{u^\alpha}{n}\frac{\partial n}{\partial x^\alpha}\right) + u^\beta \frac{\partial}{\partial x^\beta}(p+\rho)$$

$$= +n\frac{\partial}{\partial x^\alpha}\left(\frac{p+\rho}{n}\right)u^\alpha$$

$$u_\alpha \partial_\beta T^{\alpha\beta} = u^\beta\left[\frac{\partial p}{\partial x^\beta} - n\frac{\partial}{\partial x^\beta}\left(\frac{p+\rho}{n}\right)\right]$$

$$= -n u^\beta\left[p\frac{\partial}{\partial x^\beta}\left(\frac{1}{n}\right) + \frac{\partial}{\partial x^\beta}\left(\rho/n\right)\right]$$

second law of thermodynamics

$$K_B T\, d\sigma = p\, d\left(\frac{1}{n}\right) + d\left(\frac{\varrho}{n}\right)$$

$$\approx \quad T\, ds = p\, dV + dU$$

$\varrho/n \to$ energy ~~(density)~~ per particle $= \frac{\varrho/V}{N/V} = \frac{\varrho}{N}$

$1/n \to$ volume per particle $= 1/_{N/V} = V/N$

$K_B \sigma \to$ entropy per particle

$$\therefore u_\alpha \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = -K_B T \frac{\partial}{\partial x^\alpha}(n\sigma u^\alpha)$$

$$\therefore \frac{\partial}{\partial x^\alpha}(n\sigma u^\alpha) = +\frac{1}{K_B T} u_\alpha \frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta}\right)$$

$$\left(\because u_\alpha \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0 \text{ if } T^{\alpha\beta} = 0\right)$$

Let $S^\alpha = n K_B \sigma u^\alpha - \frac{1}{T} u_\beta \tau^{\alpha\beta}$

= entropy current 4-vector

$S^0 = n K_B \sigma$ = entropy density in comoving frame.

S^α is rate of entropy production per volume.

$$\therefore \frac{\partial s^\alpha}{\partial x^\alpha} = k_B \frac{\partial}{\partial x^\alpha}(n\sigma u^\alpha) + \frac{1}{T^2}\frac{\partial T}{\partial x^\alpha} u_\alpha \tau^{\alpha\beta} + \frac{1}{T}\left(\tau^{\alpha\beta}\frac{\partial u_\beta}{\partial x^\alpha} + u_\beta \frac{\partial \tau^{\alpha\beta}}{\partial x^\alpha}\right)$$

$$+ \frac{1}{T} u_\alpha \frac{\partial \tau^{\alpha\beta}}{\partial x^\beta}$$

$$\frac{\partial s^\alpha}{\partial x^\alpha} = -\frac{1}{T}\frac{\partial u_\alpha}{\partial x^\beta}\tau^{\alpha\beta} + \frac{1}{T^2}\frac{\partial T}{\partial x^\beta} u_\alpha \tau^{\alpha\beta}$$

Entropy Condition

$$\frac{\partial s^\alpha}{\partial x^\alpha} \geq 0$$

$\tau^{\alpha\beta}$ must be linear combination of velocity and temperature gradient. The inclusion of second term in s^α can be understood in this light. Without it $\partial s^\alpha / \partial x^\alpha$ is not quadratic in first derivative.

At comoving frame $u^\alpha = (u^0, u^i) = (1, 0)$

$$\Rightarrow \partial u^0 / \partial x^\alpha = \tau^{00} = 0$$

$$\frac{\partial s^\alpha}{\partial x^\alpha} = -\left(\frac{1}{T}\partial_t \vec{u}_i + \frac{1}{T^2}\frac{\partial T}{\partial x^i}\right)\tau^{i0} - \frac{1}{T}\frac{\partial u_i}{\partial x^j}\tau^{ij}$$

the choice for $\partial s^\alpha / \partial x^\alpha \geq 0$ are

$$\tau^{i0} = -\chi\left(\frac{\partial T}{\partial x^i} + T\frac{\partial u_i}{\partial t}\right)$$

$$\tau^{ij} = -\eta\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3}\nabla\cdot\vec{u}\,\delta_{ij}\right) - \xi\,\nabla\cdot\vec{u}\,\delta_{ij}$$

$$\therefore \frac{\partial s^\alpha}{\partial x^\alpha} = \frac{\chi}{T^2}\left(\nabla T + T\frac{\partial \vec{u}}{\partial t}\right)^2 + \frac{\eta}{2T}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3}\delta_{ij}\nabla\cdot\vec{u}\right)^2 + \frac{\xi}{T}(\nabla\cdot\vec{u})^2 \geq 0$$

Given $\chi \geq 0$, $\eta \geq 0$, $\xi \geq 0$

(# Note : diagonal and off-diagonal, what if $\eta = \xi$?)

Now, we need to write $T^{\alpha\beta}$ in general frame of reference.

Shear tensor

$$\sigma_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x^\beta} + \frac{\partial u_\beta}{\partial x^\alpha} - \frac{2}{3}\eta_{\alpha\beta}\frac{\partial u^\lambda}{\partial x^\lambda}$$

Heat flow vector

$$Q_\alpha = \frac{\partial T}{\partial x^\alpha} + T\frac{\partial u_\alpha}{\partial x^\beta}u^\beta$$

Projection ~~normal~~ to hyperplane normal to u^α

$$\Delta_{\alpha\beta} = \eta_{\alpha\beta} + u_\alpha u_\beta$$

We can write

$$\tau^{\alpha\beta} = -\eta\,\Delta^{\alpha\sigma}\Delta^{\beta\rho}\sigma_{\sigma\rho} - \chi\left(\Delta^{\alpha\sigma}u^\beta + \Delta^{\beta\sigma}u^\alpha\right)Q_\sigma - \xi\Delta^{\alpha\beta}\frac{\partial u^\lambda}{\partial x^\lambda}$$

$\eta \to$ shear viscosity

$\chi \to$ heat conduction

$\xi \to$ bulk viscosity

$$J^\alpha = 0$$